

Waves and Instabilities

A basic course on hydrodynamics
with geophysical applications



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Preface

These lecture notes are based on a course held at the Brandenburg Technical University Cottbus 2005/06 and 2006/07. The course consists of two hours of lectures weekly, supplemented by two hours of exercises. It has three main parts: (I) Introduction to the continuum description and to the basic hydrodynamic equations, (II) waves in inviscid fluids, and (III) instabilities and pattern formation.

Since the lectures were developed for the master course *Euro Hydro-Informatics and Water Management*, a certain emphasis is layed on geophysical applications as

- water waves in deep and shallow water
- generation of water waves by sea quakes, Tsunamis
- fluid motion in a rotating frame, Coriolis force
- atmospheric motion influenced by Coriolis force and thermal gradients
- Solitons
- Kelvin-Helmholtz instability
- Rayleigh-Bénard thermal convection

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Contents

I	Basics	7
1	Kinematics of a Continuum	9
1.1	Material description and spatial description	9
1.2	Material derivative	11
1.3	Displacement field	13
1.4	Infinitesimal deformations	15
1.5	A short paragraph on tensors	16
1.5.1	Definition as a linear transformation	16
1.5.2	Components of a tensor	17
1.5.3	Symmetric tensors	19
1.5.4	Sum of two tensors	19
1.5.5	Product of two tensors	19
1.5.6	Contraction and trace of a tensor	21
1.5.7	Antisymmetric Tensors	21
1.5.8	The dual vector	22
1.5.9	Eigenvalues and eigenvectors of a tensor	23
1.5.10	Differential operators, scalar-, vector-, tensor-fields	24
1.5.11	Defining tensors by transformation laws	26
1.6	Decomposition of the distortion tensor	27
2	Forces, Deformation and Stress	29
2.1	Stress vector and stress tensor	31
2.1.1	Stress vector	31
2.1.2	Stress tensor	31
2.1.3	Diagonal and off-diagonal components of the stress tensor	33
2.1.4	Symmetry of the stress tensor	34
2.2	Stress and forces	34
3	The Euler Equations	39
3.1	Preliminaries	39
3.2	Conservation laws and basic equations	40
3.2.1	Conserved quantities	40
3.2.2	Global conservation versus local conservation	41

3.2.3	Velocity	42
3.2.4	Density	44
3.2.5	Temperature	46
3.3	Hydrostatics	46
3.3.1	Basic equations	46
3.3.2	Examples for pressure and density distribution	47
3.4	Potential Flows and Bernoulli's Theorem	49
3.4.1	Potential flow	49
3.4.2	Receipt for solution of problems	50
3.4.3	Example: Plane (2D) flow around a cylinder	50
3.4.4	Bernoulli's theorem	51
4	The Navier-Stokes Equations	53
4.1	Stress tensor	53
4.2	Viscosities	55
4.3	The Navier-Stokes equations	57
4.4	Boundary conditions	58
4.4.1	Application	61
4.5	Stream functions	63
4.5.1	Plane flows	63
4.5.2	Axisymmetric flows	64
4.6	Viscous potential flows	65
4.7	Boundary layers	66
II	Waves	69
5	Sound Waves	71
5.1	Preliminaries	71
5.2	Sound speed	72
5.3	Wave equation for sound waves	73
5.3.1	Compression waves	74
5.3.2	State equation	75
6	Surface Waves	77
6.1	Preliminaries	77
6.2	Gravity waves	78
6.2.1	equations for flow	78
6.2.2	Equation for the location of the free surface	78
6.2.3	Basic equations and linear solutions	80
6.3	The Shallow Water equations	83
6.3.1	The linearized Shallow Water equations	85
6.3.2	Numerical solutions of the nonlinear Shallow Water equations	87

6.3.3	Shallow water waves on a modulated ground	88
6.3.4	Generation of waves by a time-dependent ground	90
6.3.5	Tsunamis	93
7	Influence of a slow rotation	97
7.1	Navier-Stokes equations in a rotating system	97
7.1.1	Concept of “Eddy Viscosity”	99
7.2	Thin layers on a Rotating Sphere	100
7.3	Geostrophic Flows	103
7.4	Thermal wind	105
7.5	The Shallow Water equations with rotation	108
7.5.1	Dispersion relation	110
7.5.2	Trajectories of volume elements	111
7.5.3	Large scale motion (inertial motion)	112
7.5.4	Waves along a coastline (Kelvin waves)	113
7.5.5	Gulf stream	115
8	Solitons	117
8.1	Discovery	117
8.2	The Korteweg – de Vries equation	119
8.3	Numerical solutions of the KdV equation	121
8.3.1	Conserved quantities	122
8.3.2	Spectral methods	123
8.3.3	Finite difference methods	124
8.3.4	Time integration	125
8.3.5	Method of Zabusky and Kruskal	127
8.4	Analytical one-soliton solution	128
III	Instabilities	133
9	Concepts	135
9.1	Exchange of stability	135
9.2	A model from Classical Mechanics	136
9.3	Linear stability analysis	137
9.3.1	One equation	138
9.3.2	General method, many equations	139
9.4	Pattern Formation	141
10	The Kelvin-Helmholtz Instability	149
10.1	The system	149
10.2	Mechanism and examples	150
10.2.1	Mechanism	150

10.2.2	Examples	151
10.3	Equations	152
10.4	The linear problem	153
10.5	Instability	155
10.6	Example: water/air – layers	157
11	The Rayleigh-Bénard Instability	159
11.1	The system	160
11.2	The basic equations	162
11.2.1	Navier-Stokes equations	162
11.2.2	Temperature equation	163
11.2.3	State equation	164
11.2.4	Basic equations in dimensionless formulation	165
11.3	The conductive state and its instability	166
11.3.1	The conductive state	166
11.3.2	Linear stability analysis	167
11.3.3	Linear growth rates and modes	168
11.4	The fully nonlinear equations – Numerical results	171
11.4.1	The basic equations in three dimensions	171
11.4.2	Boundary conditions	172
11.4.3	Results	174
	Further reading	179
A	Exercises	181
A.1	Sheet 1	181
A.2	Sheet 2	183
A.3	Sheet 3	185
A.4	Sheet 4	188
A.5	Sheet 5	191
A.6	Sheet 6	193
A.7	Sheet 7	196