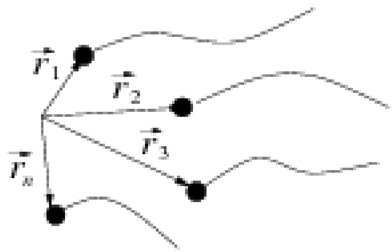
Chapter 1

Kinematics of a Continuum

1.1 Material description and spatial description

- A solid, a fluid or a gas consists of microscopic particles. We wish to describe the motion of N such particles located at $\vec{r}_n$.



complete description by

$$\begin{aligned} \ddot{\vec{r}}_n(t) &= \vec{F}_N(\vec{r}_1 \ldots \vec{r}_N) \\ n &= 1 \ldots N \end{aligned}$$

This is in Newtonian mechanics. Other possibilities are to use a quantum mechanical wave function $\underline{\Psi}(\vec{r}_1 \ldots \vec{r}_N, t)$. This can be important for quantum fluids, but will not be considered in this lectures.

- microscopic picture of the "continuum"

Problems:
- $N \approx 0(10^{23})$
- too complicated
- too much information

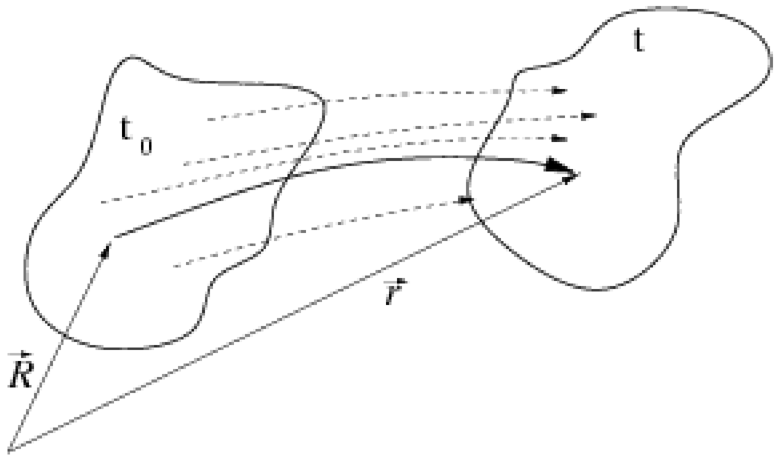
solutions: $\nearrow$ Continuum limit $\searrow$ mesoscopic theory; $\searrow$ Statistical physics $\nearrow$ mesoscopic theory

We consider the continuum limit:

Idea: take infinitely many particles
- particles are no longer countable
- description by $\vec{r}_n$ makes no sense
- instead of particles, use volume elements

a volume element contains so many particles that microscopic properties are not seen (averaging)

Identification: A volume element is identified by its position $\vec{R}$ at $t = t_0$
– Reference state, undeformed state –



$$\vec{R} = \vec{r}(\vec{R}, t = t_0)$$

Complete description, if $\vec{r}(\vec{R}, t)$ is known!

$\vec{r}(\vec{R}, t)$ is a (complicated) transformation with t as a parameter

It maps $\underbrace{\vec{R} \longrightarrow \vec{r}}_{+}$

$\vec{r}(\vec{R}, t)$ is called "material description" (Lagrangian description)

same for velocity, temperature, etc.

$$\left.\begin{array}{l}\vec{v}(\vec{R},t)\\ T(\vec{R},t)\end{array}\right\}\text{ Velocity, temperature of \underline{the same} volume element}$$

More convenient is the "spatial description" (Eulerian description)

$$\left.\begin{array}{l}T(\vec{r},t)\\ \vec{V}(\vec{r},t)\end{array}\right\}\text{ Velocity, temperature at a certain point } \vec{r} \text{ and a certain time } t\text{, but \underline{different} volume elements}$$

1.2 Material derivative

How does a certain property of a specified volume element change in course of time?

Answer is easy in material description!

e. g. temperature

$$\frac{\partial T}{\partial t} = \left(\frac{\partial T}{\partial t}\right)_R \equiv \frac{DT}{Dt}$$

take the same $\vec{R}$

follow the volume element

But what is the answer in the spatial description (fixed laboratory frame)?

$$T(\vec{r},t) = T(\vec{r}(\vec{R},t),t)$$

$\rightarrow \vec{r}$ depends on t.

using the chain rule

$$\begin{aligned}\frac{DT}{Dt} &= \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\underbrace{\frac{dx}{dt}}_{v_x} + \frac{\partial T}{\partial y}\underbrace{\frac{dy}{dt}}_{v_y} + \frac{\partial T}{\partial z}\underbrace{\frac{\partial z}{dt}}_{v_z} \\ &= \frac{\partial T}{\partial t} + (\vec{v}\cdot\nabla)T \qquad (1.1) \\ \nabla &\equiv \begin{pmatrix}\partial_x \\ \partial_y \\ \partial_z\end{pmatrix} \quad \text{Nabla vector operator}\end{aligned}$$

The time derivative of any quantity $A(\vec{r},t)$ in spatial description has always two parts

$$\underset{\substack{\uparrow \\ \text{Change of} \\ \text{A of the} \\ \text{Volume element}}}{\frac{DA}{Dt}} = \underset{\substack{\uparrow \\ \text{Change of} \\ \text{A at a} \\ \text{fixed position } \vec{r}}}{\frac{\partial A}{\partial t}} + \underset{\substack{\uparrow \\ \text{Change of} \\ \text{A while} \\ \text{moving along} \\ \vec{r}(\vec{R},t) \text{ with } \vec{v}(\vec{r},t)}}{(\vec{v}\cdot\nabla)A}$$

one needs the velocity field

$$\vec{v}(\vec{r},t)$$

if the continuum is moving.

Another important quantity is the acceleration of a certain volume element

$$\underset{\substack{\uparrow \\ \text{material} \\ \text{description}}}{\vec{a}(R,t)} = \underset{\substack{\uparrow \\ \text{spatial} \\ \text{description}}}{\frac{D\vec{v}}{Dt}} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}$$

dynamics (classical) of a continuum is fixed by Newtons 2nd law:

$$\boxed{m\left(\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right)=\vec{f}(\vec{r},t)}$$

$\vec{f}$: (inner and outer) forces at position $\vec{r}$, time t

– Fluid mechanics is (almost) always a nonlinear theory

  but for small velocities:

$$\frac{D\vec{v}}{Dt}\approx\frac{\partial\vec{v}}{\partial t}$$

– This approximation is only good close to walls or obstacles (boundary layers), or in microfluidics. For large velocities, Turbulence occurs which is clearly a non-linear phenomenon. Also instabilities and pattern formation exist due to nonlinearities.

1.3 Displacement field

up to here: description by trajectories of volume elements (like particles).

$$\vec{r}=\vec{r}(\vec{R},t)$$

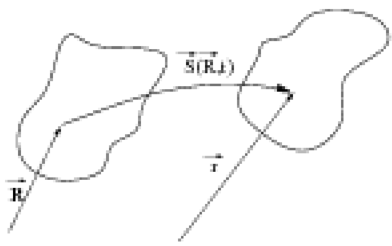


Figure 1.1:

now: consider the displacement $\vec{S}(\vec{R},t)$

– property of a certain volume element

$$\vec{r}(\vec{R},t) = \vec{R} + \vec{S}(\vec{R},t)$$

for small displacement we have $\vec{S}(\vec{R},t) = \vec{S}(\vec{r},t)$

proof:

$$\vec{S}(\vec{r},t) = \vec{S}(\vec{R}+\vec{S},t) \overset{\text{Taylor}}{=} \vec{S}(\vec{R},t) + \underbrace{\nabla\vec{S})\,|_R \cdot\vec{S}}_{0(S^2)} \tag{1.2}$$

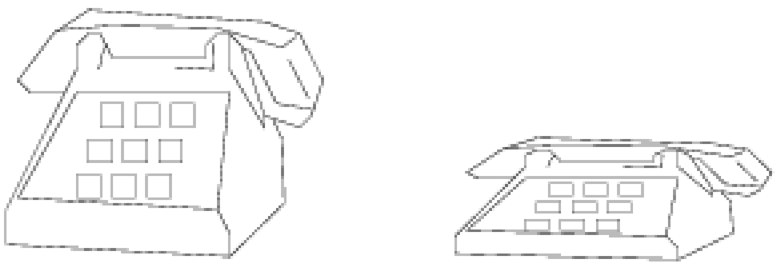
Example: uniaxial compression

$$x = X, \qquad y = Y, \qquad z = \frac{Z}{2}$$

Compute the displacement field.

Answer:

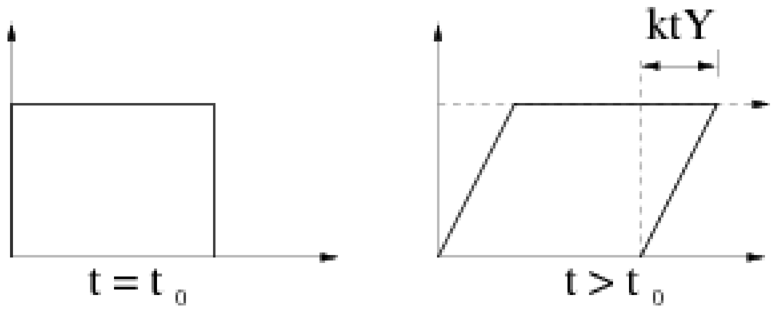
$$\vec{S} = \vec{r} - \vec{R} = \begin{pmatrix} 0 \\ 0 \\ \frac{-Z}{2} \end{pmatrix}$$



Example: shearing motion

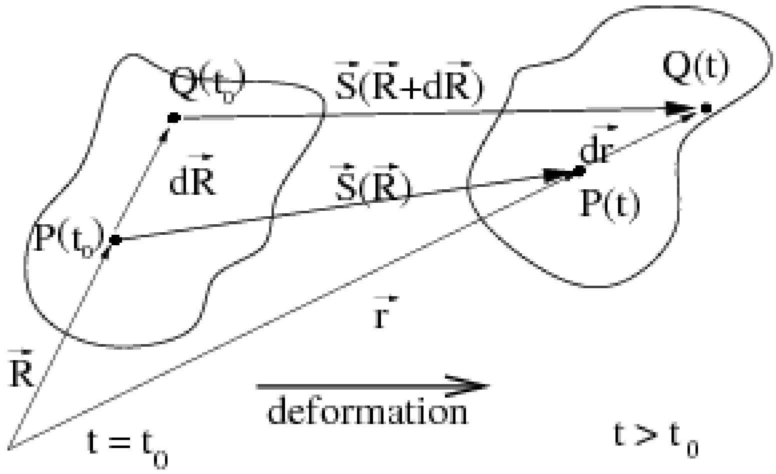
$$x = X + k\cdot t\cdot Y, \qquad y = Y, \qquad z = Z$$

$$\vec{S} = \begin{pmatrix} k\cdot t\cdot Y \\ 0 \\ 0 \end{pmatrix}$$



1.4 Infinitesimal deformations

consider two neighboring points P and Q.



$$\begin{aligned} P: \quad & \vec{r} &=& \ \vec{R}+\vec{S}(\vec{R}) \\ Q: \quad & \vec{r}+d\vec{r} &=& \ \vec{R}+d\vec{R}+\vec{S}(\vec{R}+d\vec{R}) \\ Q-P: \quad & d\vec{r} &=& \ d\vec{R}+ \underbrace{\vec{S}(\vec{R}+d\vec{R})}_{\text{Taylor} \,=\, \vec{S}(\vec{R})+d\vec{R}(\vec{\nabla}\circ\vec{S})} \end{aligned}$$

$$d\vec{r} = d\vec{R}+d\vec{R}\underbrace{(\nabla\circ\vec{S})}_{=d\vec{S}}$$

$d\vec{r}$ and $d\vec{S}$ have *different* directions $\rightarrow$ $(\nabla\circ\vec{S})$ is a *tensor*!

$$\nabla \circ \vec{S} = \begin{pmatrix} \partial_x S_x & \partial_x S_y & \partial_x S_z \\ \partial_y S_x & \partial_y S_y & \partial_y S_z \\ \partial_z S_x & \partial_z S_y & \partial_z S_z \end{pmatrix} \tag{1.3}$$

dyadic product of ∇ and $\vec{S}$ in components:

$$\frac{\partial S_j}{\partial x_i}$$

displacement gradient, distorsion tensor

$$\underline{\beta} = \nabla \circ \vec{S}\,, \qquad \beta_{ij} = \frac{\partial S_j}{\partial x_i}$$

$\underline{\beta}$ assigns to two points with distance $d\vec{r}$ the relative displacement $d\vec{S}$. It describes completely an infinitesimal deformation!

$$d\vec{S} = d\vec{r} \cdot \underline{\beta}$$

In components

$$dS_i = \sum_{j=1}^{3} \beta_{ji}\, dr_j \tag{1.4}$$

In this script we use Einstein's sum convention. This means we drop the sum and write instead of (1.4) the short form

$$dS_i = \beta_{ji}\, dr_j$$

where over indices which occur twice on one side of an equation the sum runs from one to three.

1.5 A short paragraph on tensors

1.5.1 Definition as a linear transformation

Let $\underline{T}$ be a transformation (operation) which transforms any vector into another vector:

$$\underline{T} \cdot \vec{a} = \vec{b}, \tag{1.5}$$

if $\underline{T}$ has the linear properties

$$\begin{aligned} \underline{T} \cdot (\vec{a}_1 + \vec{a}_2) &= \underline{T} \cdot \vec{a}_1 + \underline{T} \cdot \vec{a}_2, \qquad \vec{a}_1, \vec{a}_2 \qquad \text{arbitrary} \\ \underline{T} \cdot (\alpha \cdot \vec{a}) &= \alpha \cdot \underline{T} \cdot \vec{a} \qquad\qquad \alpha, \vec{a} \qquad \text{arbitrary} \end{aligned}$$

then $\underline{T}$ is a second order tensor (linear transformation).

If for all $\vec{a}$

$$\underline{T} \cdot \vec{a} = \underline{S} \cdot \vec{a}$$

then

$$\underline{T} = \underline{S}$$

1.5.2 Components of a tensor

The components of a Tensor depend on the base vectors $\hat{e}_i$.

For vectors:

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \text{is short for} \qquad \vec{a} = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3$$

Multiplying from the left with $\hat{e}_i$ yields (if $(\hat{e}_i \cdot \hat{e}_j) = \delta_{ij}$)

$$\vec{a} \cdot \hat{e}_i = a_i$$

Now we use the special orthogonal base

$$\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Writing down (1.5) in components:

$$\begin{aligned} T_{11}a_1 + T_{12}a_2 + T_{13}a_3 &= b_1 \\ T_{21}a_1 + T_{22}a_2 + T_{23}a_3 &= b_2 \\ T_{31}a_1 + T_{32}a_2 + T_{33}a_3 &= b_3 \end{aligned}$$

and using $\vec{a} = \hat{e}_1$ gives

$$\underline{T} \cdot \hat{e}_1 = \begin{pmatrix} T_{11} \\ T_{21} \\ T_{31} \end{pmatrix} = T_{11}\hat{e}_1 + T_{21}\hat{e}_2 + T_{31}\hat{e}_3 \tag{1.6}$$

or in general

$$\begin{aligned} \underline{T} \cdot \hat{e}_j &= T_{1j}\hat{e}_1 + T_{2j}\hat{e}_2 + T_{3j}\hat{e}_3 \\ &= T_{kj}\hat{e}_k \end{aligned} \tag{1.7}$$

and multiplying with $\hat{e}_i$ finally

$$\hat{e}_i \underline{T} \hat{e}_j = T_{kj} \underbrace{(\hat{e}_i \hat{e}_k)}_{\delta_{ik}} = T_{ij} \tag{1.8}$$

In the same way as

$$a_i = \vec{a} \cdot \hat{e}_i$$

we may write

$$T_{ij} = \hat{e}_i \cdot T_{ij} \cdot \hat{e}_j$$

T_{ij} is called “the matrix of the tensor $\underline{T}$”.

but remember: with respect to the base vectors $\hat{e}_i$

1.5.3 Symmetric tensors

If

$$\underline{T} \cdot \vec{a} = \vec{a} \cdot \underline{T} \quad \text{then} \quad T_{ij} = T_{ji}$$

and $\underline{T}$ is a <u>symmetric</u> tensor.

1.5.4 Sum of two tensors

if

$$\underline{T} \cdot \vec{a} + \underline{S} \cdot \vec{a} = \underline{W} \cdot \vec{a}$$

for all $\vec{a}$, then

$$\underline{W} = \underline{T} + \underline{S}$$

is the sum of $\underline{T}$ and $\underline{S}$

in components

$$W_{ij} = T_{ij} + S_{ij}$$

1.5.5 Product of two tensors

(a) Inner product

$$\begin{aligned} \text{Let} \quad (\underline{T} \cdot \underline{S}) \cdot \vec{a} &= \underline{T} \cdot (\underline{S} \cdot \vec{a}) \\ \text{and} \quad (\underline{S} \cdot \underline{T}) \cdot \vec{a} &= \underline{S} \cdot (\underline{T} \cdot \vec{a}), \end{aligned}$$

for arbitrary $\vec{a}$, then $(\underline{T} \cdot \underline{S})$ and $(\underline{S} \cdot \underline{T})$ are called (inner) product of $\underline{T}$ and $\underline{S}$ (Obviously there are two possibilities).

The components of the inner product read:

$$
\begin{aligned}
(\underline{T}\cdot\underline{S})_{ij} &= \hat{e}_i\cdot(\underline{T}\cdot\underline{S})\cdot\hat{e}_j = \hat{e}_i\cdot\underline{T}(\underbrace{\underline{S}\cdot\hat{e}_j}_{=S_{kj}\cdot\hat{e}_k}) \\
&= \underbrace{(\hat{e}_i\cdot\underline{T}\cdot\hat{e}_k)}_{T_{ik}}\cdot S_{kj} \\
&= T_{ik}\,S_{kj}
\end{aligned} \tag{1.9}
$$

and similarly

$$
(\underline{S}\cdot\underline{T})_{ij} = S_{ik}T_{kj} \tag{1.10}
$$

in general:

$$
\underline{S}\cdot\underline{T} \neq \underline{T}\cdot\underline{S}
$$

the tensor product is not commutative!

(compare to inner (scalar) product of two vectors (scalar valued = independent on $\hat{e}_i$)).

$$
\begin{aligned}
\vec{a}\cdot\vec{b} &= (a_i\cdot\hat{e}_i)\,\left(b_j\cdot\hat{e}_j\right) = a_i b_j \underbrace{(\hat{e}_i\cdot\hat{e}_j)}_{\delta_{ij}} \\
&= a_i\,b_i
\end{aligned}
$$

(b) outer product

Def:

$$
T_{ij}S_{kl} = Q_{ijkl}
$$

is the outer product of $\underline{T}$ and $\underline{S}$. Then, $\underline{Q}$ is a fourth order tensor

Compare to outer product of two vectors!

$$
a_i \circ b_j = T_{ij}, \qquad \vec{a}\circ\vec{b} = \underline{T}
$$

$\underline{T}$ is a tensor (second order).

→ "dyadic product"

1.5.6 Contraction and trace of a tensor

The contraction of a tensor is the sum over two indices. This reduces its order by two.

Example:

$$\sum_j Q_{ijjm} = Q_{ijjm} = P_{im}$$

$\underline{P}$ is the (one of the) contraction (s) of $\underline{Q}$

The trace of a tensor is the contraction of a second order tensor, what leaves a scalar.

$$\begin{aligned} T_{ii} &= \mathrm{tr}(\underline{T}), \qquad \text{trace of } \underline{T} \\ \mathrm{tr}(\underline{T}+\underline{S}) &= \mathrm{tr}(\underline{T})+\mathrm{tr}(\underline{S}) \\ \mathrm{tr}(\vec{a}\circ\vec{b}) &= \vec{a}\cdot\vec{b} \end{aligned}$$

1.5.7 Antisymmetric Tensors

$$\begin{array}{llll} \text{if} \quad T_{ij} = T_{ji} & \text{then} & \underline{T} & \text{is a symmetric tensor} \\ \text{if} \quad T_{ij} = -T_{ji} & \text{then} & \underline{T} & \text{is an antisymmetric tensor} \end{array}$$

antisymmetric tensors of 2nd order can have only 3 independent elements

$$\underline{T}^A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

and clearly

$$T^A_{ii} = 0, \qquad \text{and} \quad \mathrm{tr}\left(\underline{T}^A\right) = 0$$

Important third order antisymmetric tensor: ε tensor

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } i,j,k = 1,2,3 \text{ or even permutations} \\ -1 & \text{if } i,j,k \text{ are odd permutations of 1, 2, 3} \\ 0 & \text{if 2 or 3 indices are equal} \end{cases}$$

$\underline{\varepsilon}$ can be used to compute the vector product (don't confuse with the outer product)

$$\vec{a} = \vec{b} \times \vec{c} \qquad \Rightarrow \qquad a_i = \varepsilon_{ijk} b_j c_k$$

Each tensor can be decomposed:

$$\underline{T} = \underline{T}^S + \underline{T}^A$$

with the symmetric tensor

$$\underline{T}^S = \frac{1}{2}(\underline{T} + \underline{T}^T)$$

and the antisymmetric tensor

$$\underline{T}^A = \frac{1}{2}(\underline{T} - \underline{T}^T)$$

$\underline{T}^T$ denotes the transpose of $\underline{T}$:

$$T^T_{ij} = T_{ji}$$

1.5.8 The dual vector

Let $\underline{T}^A$ be an antisymmetric 2nd order tensor. Then its dual vector is defined as

$$\vec{t}^A = \begin{pmatrix} -T^A_{23} \\ T^A_{31} \\ T^A_{12} \end{pmatrix} = -\frac{1}{2}\,\varepsilon_{ijk}\,T^A_{jk}\,\hat{e}_i \tag{1.11}$$

and

$$\underline{T}^A \cdot \vec{a} = \vec{t}^A \times \vec{a}, \qquad \perp \text{ on } \vec{a} \text{ and } \vec{t}^A$$

can be used to compute the vector product.

1.5.9 Eigenvalues and eigenvectors of a tensor

Definition:

If

$$\underline{T} \cdot \vec{a} = \lambda\, \vec{a}, \qquad \lambda \in \mathcal{C}$$

then $\vec{a}$ is an eigenvector of $\underline{T}$

λ is the eigenvalue of $\underline{T}$ that belongs to $\vec{a}$

To compute the eigenvalues and eigenvectors one has to solve a linear homogeneous system

$$(\underline{T} - \underline{1} \cdot \lambda) \cdot \vec{a} = 0$$

This has nontrivial solutions ($\vec{a} \neq 0$), only if the solvability condition

$$\det\,(\underline{T} - \underline{1}\lambda) = 0$$

is fulfilled. This leads to the characteristic equation (polynomial in λ)

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

It has three (in general complex) roots:

$$\lambda_1,\ \lambda_2,\ \lambda_3$$

From that one may compute the eigenvectors

$$\vec{a}_1,\ \vec{a}_2,\ \vec{a}_3$$

A symmetric tensor has three real valued eigenvalues. Its eigenvectors $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ are mutually orthogonal. They can be normalized, $|\vec{a}_i| = 1$ and form a system of orthogonal base vectors

$$\hat{a}_i \cdot \hat{a}_j = \delta_{ij}$$

λ_i are independent from the base $\hat{e}_i$. From that it follows that I_1, I_2, I_3 are also independent.

They are called scalar invariants of $\underline{T}$

$$I_1 = T_{11} + T_{22} + T_{33} = \mathrm{tr}\,(\underline{T}) \tag{1.12}$$

$$I_2 = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix}$$

$$= \frac{1}{2}\left[(\mathrm{tr}\,(\underline{T}))^2 - tr(\underline{T}^2)\right] \tag{1.13}$$

$$I_3 = \det\,(\underline{T}) \tag{1.14}$$

or, in terms of λ_i

$$\begin{aligned} I_1 &= \lambda_1 + \lambda_2 + \lambda_3 \\ I_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\ I_3 &= \lambda_1\lambda_2\lambda_3 \end{aligned}$$

1.5.10 Differential operators, scalar-, vector-, tensor-fields

(a) Gradient of a scalar field $\Phi(\vec{r})$

How changes Φ if $\vec{r}$ is changed?

$$\underbrace{d\Phi}_{\text{scalar}} = \Phi(\vec{r} + d\vec{r}) - \Phi(\vec{r}) \equiv \underbrace{\nabla\Phi}_{\text{vector}} \cdot \underbrace{d\vec{r}}_{\text{vector}} \tag{1.15}$$

Taylor expansion:

$$\begin{aligned}
\Phi(\vec{r}+d\vec{r})-\Phi(\vec{r}) &= \partial_x\Phi dx+\partial_y\Phi dy+\partial_z\Phi dz+O(dr^2) \\
&= \nabla\Phi\cdot d\vec{r} \\
\rightarrow \quad \nabla\Phi=\mathrm{grad}\Phi &= \begin{pmatrix} \partial_x\Phi \\ \partial_y\Phi \\ \partial_z\Phi \end{pmatrix}
\end{aligned} \tag{1.16}$$

directional derivative:

$$\underbrace{\frac{\partial\Phi}{\partial\vec{n}}}_{\text{scalar,}} = \vec{n}\cdot\nabla\Phi \tag{1.17}$$

This is a scalar quantity which measures the change of Φ if $\vec{r}$ goes to $\vec{r}+d\vec{r}$ and $d\vec{r} \parallel \vec{n}$

(b) Gradient of a vector field $\vec{V}(\vec{r})$ (outer product)

$$\underbrace{d\vec{V}}_{\text{vector}} = \vec{V}(\vec{r}+d\vec{r})-\vec{V}(\vec{r}) \equiv \underbrace{d\vec{r}}_{\text{vector}} \cdot \underbrace{(\nabla\circ\vec{V})}_{\text{tensor}} \tag{1.18}$$

in general: $d\vec{V}$ and $d\vec{r}$ are not parallel!

Taylor expansion (in components)

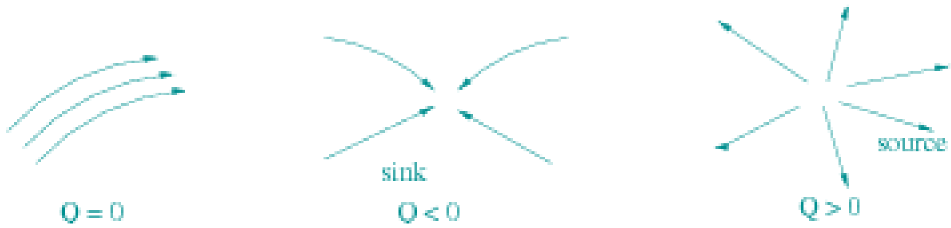
$$\begin{aligned}
V_i(\vec{r}+d\vec{r})-V_i(\vec{r}) &= \partial_x V_i dx+\partial_y V_i dy+\partial_z V_i dz \\
&= \frac{\partial V_i}{\partial x_j}dx_j \\
(\nabla\circ\vec{V})_{ij} &= \frac{\partial V_j}{\partial x_i} \qquad \text{dyadic product of } \nabla \text{ and } \vec{V}
\end{aligned} \tag{1.19}$$

(c) Divergence of a vector field (inner product)

scalar product of $\nabla, \vec{V}$:

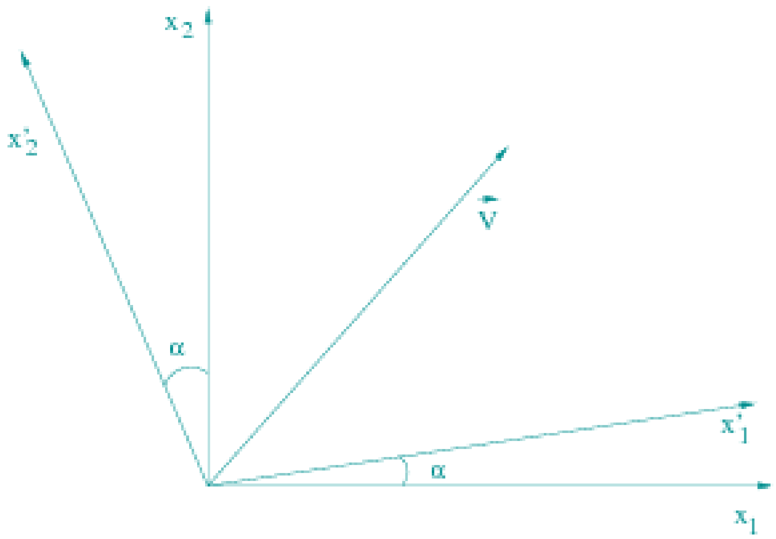
$$\nabla\cdot\vec{V} \equiv \mathrm{div}\,\vec{V} = \frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z} = \underbrace{Q(\vec{r})}_{\text{scalar}} \tag{1.20}$$

$Q(\vec{r})$ denotes sources and sinks of $\vec{V}(\vec{r})$!



1.5.11 Defining tensors by transformation laws

The laws of nature are independent of (the orientation of) the coordinate systems.



transformation

$$
\begin{aligned}
x_i &= x_i(x_1', x_2', x_3') \\
x_i' &= x_i'(x_1, x_2, x_3)
\end{aligned}
$$

The Jacobi matrix is defined as

$$A_{ij} = \frac{\partial x_i}{\partial x_j'}$$

Physical objects are transformed by certain rules, depending on their vector character:

1. Scalars (mass, charge, density, temperature, ...)

$$S(x_i) = S'(x'_i)$$

2. Vectors (forces, velocity, acceleration, heat flow, ...)

$$V_i = A_{ij} V'_j$$

3. Tensors (stress, deformation, ...)

$$T_{ij} = A_{ik} A_{jl} T'_{kl}$$

1.6 Decomposition of the distortion tensor

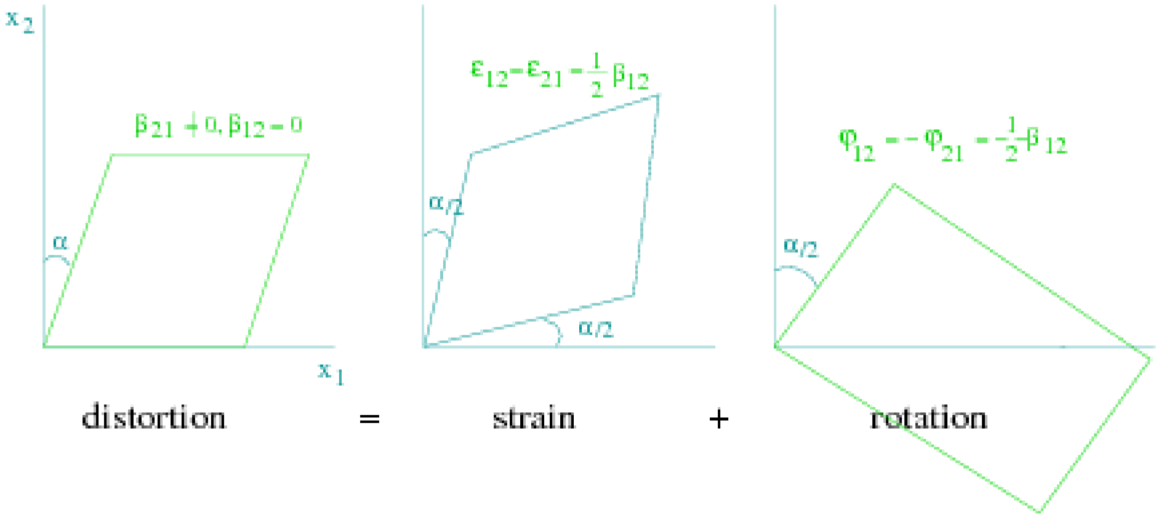
We first write the distortion tensor as a sum of a symmetric and an antisymmetric tensor and then interpret the physical meaning of the two parts.

$$(\nabla \circ \vec{S}) = \underline{\beta} = \underline{\varepsilon} + \underline{\varphi} \qquad (1.21)$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial S_j}{\partial x_j} + \frac{\partial S_i}{\partial x_j}\right) \quad \text{symmetric strain tensor} \qquad (1.22)$$

$$\varphi_{ij} = \frac{1}{2}\left(\frac{\partial S_j}{\partial x_j} - \frac{\partial S_j}{\partial x_j}\right) \quad \text{antisymmetric rotation tensor} \qquad (1.23)$$

– Geometrical interpretation of the decomposition



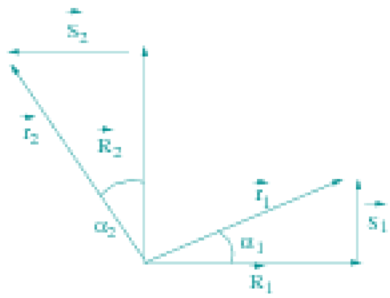
(a) Symmetric part

$\underline{\varepsilon}$ describes volume dilatation (compression) and shear

Change of volume:

$$\frac{\Delta V}{V} = \mathrm{tr}\,\underline{\beta} = \mathrm{tr}\,\underline{\varepsilon} = \mathrm{div}\,\vec{s} \tag{1.24}$$

(b) Antisymmetric part



$S_{1y} = \alpha_1 X$

$S_{2x} = -\alpha_2 Y$

pure rotation: $\alpha_1 = \alpha_2 = \alpha$

$$\begin{aligned}
\beta_{xy} &= \frac{\partial S_y}{\partial x} = \alpha_1 = \alpha \\
\beta_{yx} &= \frac{\partial S_x}{\partial y} = -\alpha_2 = -\alpha
\end{aligned} \tag{1.25}$$

$$\begin{aligned}
\varphi_{xy} &= \frac{1}{2}\left(\frac{\partial S_y}{\partial x} - \frac{\partial S_x}{\partial y}\right) = \alpha \\
\varphi_{yx} &= -\varphi_{xy} = -\alpha
\end{aligned} \tag{1.26}$$

The components of the antisymmetric part are the angles of (infinite) rotations:

1. φ_{xy}: rotation with respect to z-axis
2. φ_{yz}: rotation with respect to x-axis
3. φ_{zx}: rotation with respect to y-axis