Chapter 4

The Navier-Stokes Equations

- the Navier-Stokes eqs. describe the motion of real fluids (liquid and gas) with *viscosity* under external pressure and forces.
- firstly stated as a model by C. L. Navier (1827)
- later derived systematically from continuum mechanics by G. G. Stokes (1845)

4.1 Stress tensor

- for ideal fluids, we found in the previous chapter

$$T_{ij} = T_{ij}^{E} = -p\,\delta_{ij}$$

- for viscous fluids, we must add shear stress

$$\underline{T}^{NS} = \underline{T}^{E} + \underline{\sigma}$$

$\underline{\sigma}$: viscous stress tensor

$\mathrm{div}\underline{\sigma}$: viscous forces, friction

$\rightsquigarrow$ To arrive at the Navier-Stokes eqs., we have to add $\mathrm{div}\underline{\sigma}$ on the R.H.S. of the Euler equations

How to find $\underline{\sigma}$?

Newton: if a fluid moves homogeneously, there can be no viscous stress.

$$\rightsquigarrow \qquad \underline{\sigma} = \underline{\sigma}(\partial_i v_j)$$

for most fluids, $\underline{\sigma}$ depends linearly on $\partial_i v_j$

- there are exceptions, called Non-Newtonian-Fluids, like toothpaste, mayonnaise, blood, colors (plastic fluids), pseudo-plastic fluids, etc.

Here we shall consider only **Newtonian fluids**

There are three different possibilities to construct a tensor $\sim \partial_i v_j$ if the system is isotropic. The general form reads

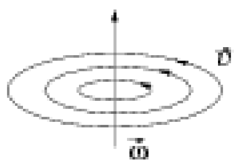
$$\underline{\sigma} = a\,(\nabla \circ \vec{v}) + b\,(\vec{v} \circ \nabla) + c\,\underline{1} \cdot \operatorname{div}\vec{v}$$

or in components

$$\sigma_{ij} = a\,\partial_i v_j + b\,\partial_j v_i + c\,\delta_{ij} \sum_\ell \partial_\ell v_\ell \tag{4.1}$$

a, b, c are material constants and related to viscosity.

In fact, only two of the three constants are independent. This can be seen considering a uniformly rotating fluid around the z-axis. In such a situation, the fluid is at rest in the rotating frame of references and no friction occurs.



Inserting

$$\vec{v} = \vec{\omega} \times \vec{r}$$

into (4.1) gives after a little algebra (try it in components and use the ε-tensor) the condition

$$a = b$$

for $\underline{\sigma} = 0$

Inserting this into (4.1) yields for the viscous stress tensor

$$\underline{\sigma} = 2a\,\underline{D} + c\,\operatorname{tr}(\underline{D}) \cdot \underline{1}$$

with

$$\underline{D} = \frac{1}{2}(\nabla \circ \vec{v} + \vec{v} \circ \nabla) \qquad \text{"rate of deformation"} \tag{4.2}$$

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \qquad \underline{D} = \dot{\underline{\varepsilon}} \qquad (\text{see } 1.6) \tag{4.3}$$

$$\text{tr}\,(\underline{D}) = D_{11} + D_{22} + D_{33} = \text{div}\,\vec{v} \tag{4.4}$$

4.2 Viscosities

Usually, the viscous stress tensor is written in another form. After some manipulations, one can decompose it in a trace free part (pure shearing, no volume change) and a diagonal part (only volume changes):

$$\begin{aligned}
\underline{\sigma} &= 2a\,\underline{D} + c\,\text{tr}\,(\underline{D}) \cdot \underline{1} \\
&= \underbrace{2a\,\underline{D} - \frac{2a}{3}\text{tr}\,(\underline{D}) \cdot \underline{1}}_{\text{tr}\,(\ldots)=0} + \underbrace{\frac{2a}{3}\text{tr}\,(\underline{D}) \cdot \underline{1} + c\,\text{tr}\,(\underline{D}) \cdot \underline{1}}_{\text{diagonal tensor}} \\
&= 2 \underbrace{a}_{\eta,\ 1.\ \text{viscosity}} \left(\underline{D} - \frac{1}{3}\text{tr}\,(\underline{D}) \cdot \underline{1}\right) + \underbrace{\left(\frac{2a}{3} + c\right)}_{\zeta,\ 2.\ \text{viscosity}} \text{tr}\,(\underline{D}) \cdot \underline{1}
\end{aligned} \tag{4.5}$$

$$\underline{\sigma} = 2\eta \left(\underline{D} - \frac{1}{3}\text{tr}\ (\underline{D}) \cdot \underline{1}\right) + \zeta\ \text{tr}\,\underline{D} \cdot \underline{1} \tag{4.6}$$

or in components

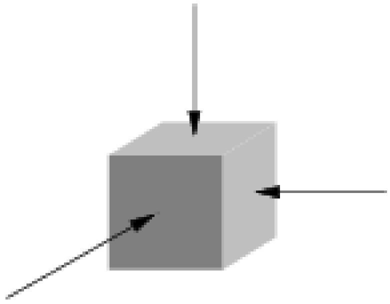
$$\sigma_{ij} = 2\eta \left(D_{ij} - \frac{1}{3}\sum_{\ell} D_{\ell\ell} \cdot \delta_{ij}\right) + \zeta \sum_{\ell} D_{\ell\ell} \cdot \delta_{ij} \tag{4.7}$$

viscosity, 1st viscosity η

η: (dynamic) viscosity, $\qquad \nu = \eta/\rho$: kinematic viscosity

$$\text{shear stress} = \eta \cdot \text{ share rate} \quad (\eta,\ \nu \hat{=} \text{ shear modulus})$$

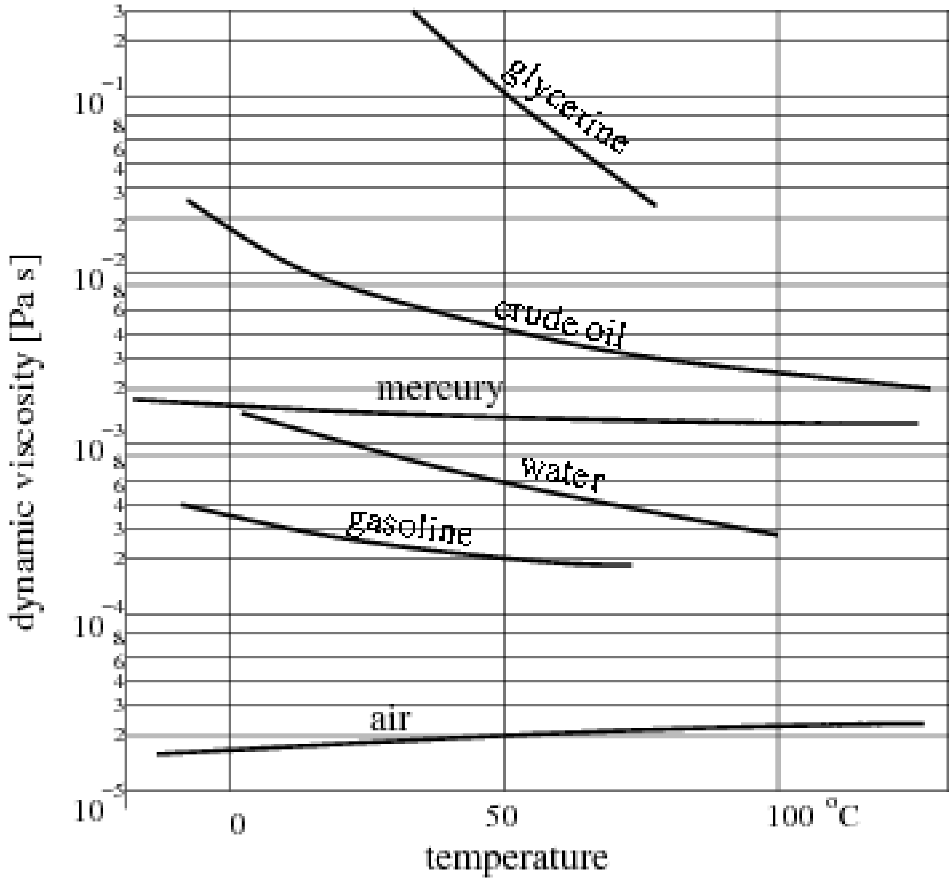
2nd viscosity ζ



$$\mathrm{tr}\,\underline{\sigma} = 3\zeta\,\mathrm{tr}\,\underline{D} = 3\zeta\,\mathrm{div}\,\vec{v} = 3\zeta\,\dot{\theta}, \qquad (4.8)$$

θ: volume dilatation, $\dot{\theta}$: rate of dilatation

dilatation stress = $\zeta \cdot$ dilatation rate ($\zeta \,\hat{=}\,$ bulk modulus)



material	dyn. v. (η)	kin. v. (ν)	material	dyn. v. (η)	kin. v. (ν)
water	1.0	1.0	petroleum	0.65	0.76
mercury	1.5	0.11	pentane	0.23	0.37
alcohol	1.8	2.2	blood (37^O)	4 - 25	4 - 25
glycerine	850	650	honey	10.000	7.000
glass	10^{23}	$0.4 \cdot 10^{23}$	melted polymers	$10^3 - 10^6$	$10^3 - 10^6$
oxygen	0.019	13	argon	0.021	12
neon	0.03	33	helium	0.019	105
hydrogen	0.008	90	grape juice	2 - 5	2 - 5
hexane	0.32	0.5	olive oil	100	110
nitrate	0.017	14	air	0.018	14

dynamic viscosities in mPa·s or 10^{-3}kg/ms, kinematic viscosities in 10^{-6}m^2/s or centi-stokes. All values at room temperature.

4.3 The Navier-Stokes equations

The total balance of forces, including viscosities, now reads

$$\rho \frac{d\vec{v}}{dt} = \operatorname{div} \underline{T} + \vec{f} = -\operatorname{grad} p + \operatorname{div} \underline{\sigma} + \vec{f} \tag{4.9}$$

now we compute div$\underline{\sigma}$:

$$(\operatorname{div} \underline{\sigma})_j = \partial_i \sigma_{ij} = \eta \, \Delta v_j + (\zeta + \frac{1}{3}\eta) \, \partial_j \operatorname{div} \vec{v} \tag{4.10}$$

assume η, ζ to be constant in space (this must not be the case if the fluid is not isothermal or consists of a spatially inhomogeneous mixture of different components):

$$\begin{aligned} \rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right) &= -\operatorname{grad} p + \vec{f} + \eta \Delta \vec{v} + \left(\xi + \tfrac{1}{3}\eta\right) \operatorname{grad} \operatorname{div} \vec{v} \\ \dot{\rho} &= -\operatorname{div} (\vec{v}\rho) \end{aligned} \tag{4.11}$$

These are the basic equations for compressible (pure, isothermal) fluids or gases. They have to be completed by material laws. The subject of compressible liquids is sometimes called "aero-dynamics".

For incompressible fluids (usually liquids or gases under high pressure), they simplify to:

$$\boxed{\begin{aligned}\rho_0\left(\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right)&=-\operatorname{grad}p+\vec{f}+\eta\Delta\vec{v}\\ \operatorname{div}\vec{v}&=0\end{aligned}} \tag{4.12}$$

The subject of incompressible fluids is called "hydrodynamics".

4.4 Boundary conditions

1. No slip conditions.

   - rigid, impermeable boundaries

viscous fluids: The fluid adheres to the wall → <u>all</u> velocity components have to be zero there!

$$\vec{v}=0 \qquad \text{at the walls}$$

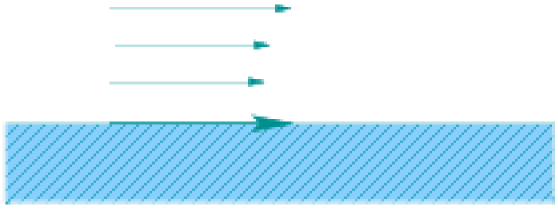
Incompressible fluids: (wall at $z = 0$)

$$\operatorname{div}\vec{v}=\underbrace{\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}}_{=0}+\frac{\partial v_z}{\partial z}=0 \tag{4.13}$$

$$\boxed{v_z=0} \qquad \text{and} \qquad \boxed{\frac{\partial v_z}{\partial z}=0}$$

In a fluid without friction (Euler eq.) there is only <u>one</u> condition:

Impermeable wall (no flux)



$$\boxed{v_z = 0}$$

Reason: Navier-Stokes eq. includes higher derivatives ($\nu\Delta\vec{v}$)

- The approximation

$$\text{Navier-Stokes} \longrightarrow \text{Euler}$$

$$\nu \rightarrow 0$$

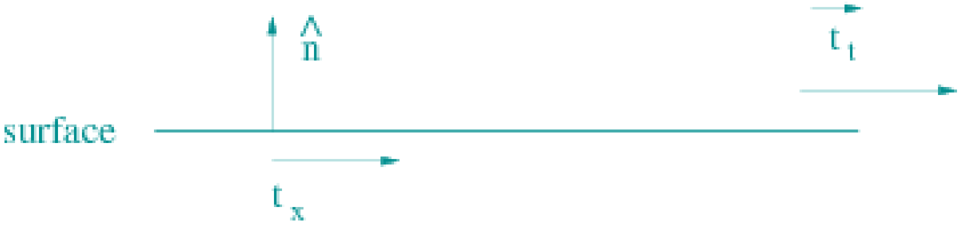
is not systematic and generally not valid!

2. Free surface condition (no flux)

viscous fluids and Euler fluids: $\boxed{v_z = 0}$

In viscous fluids, the total stress along the surface must balance.

1. constant surface tension



$$
\begin{aligned}
\hat{t}_i \cdot \vec{t}_t &= \hat{n} \cdot \underline{\sigma} \cdot \hat{t}_i = 0 \\
\hat{n} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \hat{t}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \hat{t}_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad (4.14) \\
\rightarrow \qquad \sigma_{xz} &= 0, \qquad \sigma_{yz} = 0 \qquad \text{at the free surface}
\end{aligned}
$$

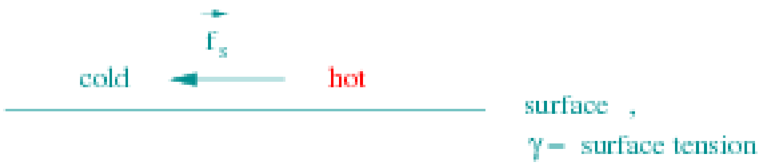
incompressible fluids

$$
\frac{\partial}{\partial z} \Bigg| \qquad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial_y} + \frac{\partial v_z}{\partial z} = 0 \qquad (4.15)
$$

$$
\frac{\partial}{\partial x} \underbrace{\frac{\partial v_x}{\partial z}}_{=0} + \frac{\partial}{\partial y} \underbrace{\frac{\partial v_y}{\partial z}}_{=0} + \frac{\partial^2 v_z}{\partial z^2} = 0 \quad |z = 0 \qquad (4.16)
$$

$$
\frac{\partial^2 v_z}{\partial z^2} = 0
$$

2. non constant surface tension (e.g. due to temperature variations)



force (stress) at surface: $\vec{f}_s = \nabla\gamma$

$$
\hat{n} \cdot \underline{\sigma} \cdot \hat{t}_i = \hat{t}_i \cdot \nabla\gamma, \qquad \hat{t}_i = \hat{t}_x, \hat{t}_y
$$

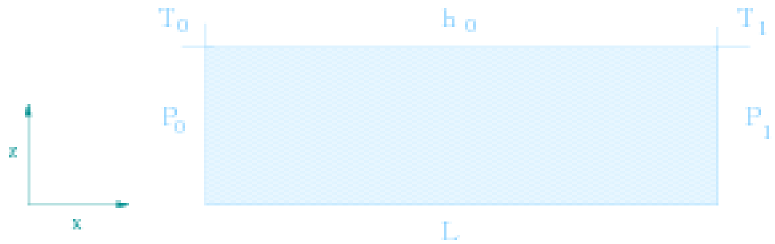
$$
\eta \frac{\partial v_x}{\partial z} = \frac{\partial \gamma}{\partial x}, \qquad \eta \frac{\partial v_y}{\partial z} = \frac{\partial \gamma}{\partial y} \qquad (4.17)
$$

or (with continuity eq.)

$$
\eta \frac{\partial^2 v_z}{\partial z^2} = -\Delta_2 \gamma
$$

4.4.1 Application

Compute the stationary flow in an open layer which is laterally heated. Assume a horizontal pressure difference δp and a horizontal temperature difference δT. Which δT is needed to keep the fluid in rest at the free surface $z = h$?



$$\delta p = P_1 - P_0$$
$$\delta T = T_1 - T_0$$

$$\vec{v} = \begin{pmatrix} v_x(z) \\ 0 \\ 0 \end{pmatrix}, \qquad \mathrm{div}\,\vec{v} = 0 \quad \text{is fulfilled} \tag{4.18}$$

Navier-Stokes eqs. (for a stationary flow all time derivatives have to vanish):

$$\rho(\underbrace{\dot{v}_x}_{=0} + v_x \underbrace{\frac{\partial v_x}{\partial x}}_{=0} + v_y \underbrace{\frac{\partial v_x}{\partial y}}_{=0} + \underbrace{v_z}_{=0} \frac{\partial v_x}{\partial z}) = \eta\,\Delta v_x - \frac{\delta p}{L} \tag{4.19}$$

$$\eta\,\frac{\partial^2 v_x}{\partial z} - \frac{\delta p}{L} = 0 \qquad \longrightarrow \qquad v_x(z) = a + bz + \frac{1}{2\eta}\frac{\delta p}{L}z^2$$

The coefficients a, b can be computed from the boundary conditions

$$v_x(0) = 0 \quad \rightsquigarrow \quad a = 0$$

$$\left.\frac{dv_x}{dz}\right|_h = \frac{1}{\eta}\cdot\frac{d\gamma}{dx} \tag{4.20}$$

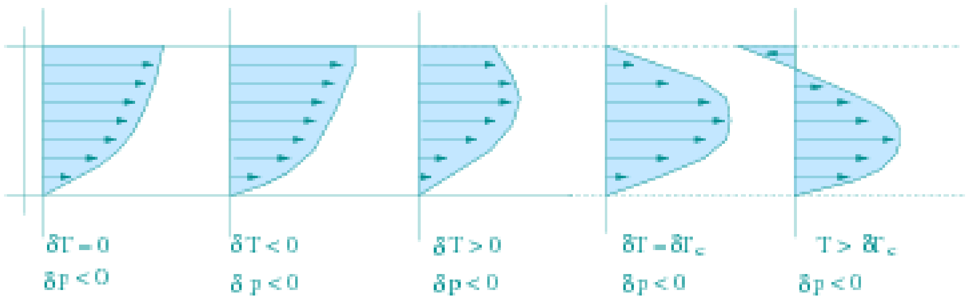
assume that the surface tension is a linear function of temperature:

$$\gamma = c_0 - c_T(T - T_0), \qquad c_T = -\left(\frac{\partial\gamma}{\partial T}\right)_{T_0} \tag{4.21}$$

$$\frac{d\gamma}{dx} = -c_T\frac{dT}{dx} = -c_T\frac{\delta T}{L} \tag{4.22}$$

Inserting this into (4.20) leads tp

$$\begin{aligned}
b + \frac{1}{\eta}\frac{\delta p}{L}h &= -\frac{c_T}{\eta}\frac{\delta T}{L} \\
b &= -\frac{1}{\eta L}\left(h \cdot \delta p + c_T \cdot \delta T\right) \\
v_x(z) &= \frac{1}{\eta L}\left[\frac{1}{2}\, z^2 \delta p - \left(h \cdot \delta p + c_T \delta T\right) z\right]
\end{aligned} \tag{4.23}$$



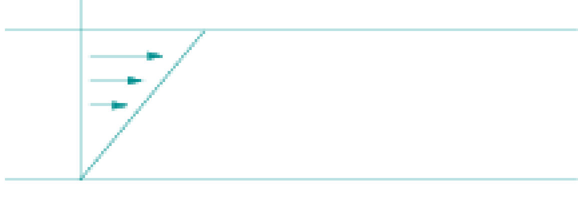
The critical temperature where the fluid is in rest at the surface follows from:

$$v_x(z = h) = 0$$

$$\boxed{\delta T_c = -\frac{1}{2}\frac{h}{c_T}\delta p} \tag{4.24}$$

for $\delta p = 0$ we have a pure shear flow

$$v_x = \frac{-c_T \delta T}{\eta L}\, z \tag{4.25}$$



4.5 Stream functions

We consider incompressible flows $\operatorname{div}\vec{v} = 0$.

4.5.1 Plane flows

Plane flow (two-dimensional):

$$v_x = v_x(x,y), \quad v_y = v_y(x,y), \quad v_z = 0$$

The stream function is a function whose contour lines are equal to the stream lines of the flow (at a fixed moment). The stream lines are computed by

$$dx = v_x dt, \qquad dy = v_y dt$$

or, eliminating dt:

$$v_x dy - v_y dx = 0 \tag{4.26}$$

The total differential of a function $\Psi(x,y)$ reads

$$d\Psi = \frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} dy$$

If the stream lines are contour lines of Ψ, $d\Psi = 0$ along the streamlines. Comparing with (4.26) yields

$$v_x = f(x,y)\,\frac{\partial \Psi}{\partial y}, \qquad v_y = -f(x,y)\,\frac{\partial \Psi}{\partial x}$$

with an arbitrary function f. This can be determined by the continuity equation

$$\operatorname{div}\vec{v} = \frac{\partial}{\partial x}\left(f\frac{\partial \Psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(f\frac{\partial \Psi}{\partial x}\right) = 0$$

Evaluating the derivatives gives

$$\frac{\partial f}{\partial x}\frac{\partial \Psi}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial \Psi}{\partial x} = 0$$

For arbitrary Ψ this can only be solved if

$$\frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0, \qquad \text{or} \qquad f = \text{const}$$

Thus we can put $f = 1$ and find

$$\begin{cases} v_x &= \dfrac{\partial \Psi}{\partial y} \\ v_y &= -\dfrac{\partial \Psi}{\partial x} \end{cases}$$

4.5.2 Axisymmetric flows

The same procedure can be applied for a flow which is symmetric with respect to a certain axis. We put the z-axis equal to that symmetry axis. Then the flow is best described using cylindrical or spherical coordinates.

Cylindrical coordinates

One has

$$v_r = v_r(r,z), \quad v_z = v_z(r,z), \quad v_\varphi = 0$$

From

$$v_r dz - v_z dr = 0 \tag{4.27}$$

one finds

$$v_r = f(r,z)\,\frac{\partial \Psi}{\partial z}, \qquad v_z = -f(r,z)\,\frac{\partial \Psi}{\partial r}$$

Inserting this into the continuity equation gives the two conditions for f

$$f + r\,\frac{\partial f}{\partial r} = 0, \qquad \frac{\partial f}{\partial z} = 0$$

which is solved by

$$f = f(r) = \frac{1}{r}$$

Finally

$$\begin{cases} v_r &= \dfrac{1}{r}\dfrac{\partial \Psi}{\partial z} \\ v_z &= -\dfrac{1}{r}\dfrac{\partial \Psi}{\partial r} \end{cases}$$

Spherical coordinates

Starting from

$$v_r = v_r(r,\vartheta), \quad v_\vartheta = v_\vartheta(r,\vartheta), \quad v_\varphi = 0$$

the stream lines are now computed by

$$dr = v_r dt, \qquad r\,d\vartheta = v_\vartheta dt$$

and eliminating dt:

$$v_r d\vartheta - \frac{v_\vartheta}{r}\, dr = 0 \tag{4.28}$$

One has

$$v_r = f(r,\vartheta)\,\frac{\partial\Psi}{\partial\vartheta}, \qquad v_\vartheta = -r\, f(r,\vartheta)\,\frac{\partial\Psi}{\partial r}$$

Inserting this into the continuity equation gives the two conditions for f

$$f\,\cot\vartheta + \frac{\partial f}{\partial\vartheta} = 0, \qquad 2f + r\,\frac{\partial f}{\partial r} = 0$$

This can be solved by a product

$$f(r,\vartheta) = h(r)\cdot g(\vartheta)$$

and gives

$$h(r) = \frac{1}{r^2}, \qquad g(\vartheta) = \frac{1}{\sin\vartheta}$$

and

$$\begin{cases} v_r &= \dfrac{1}{r^2\sin\vartheta}\dfrac{\partial\Psi}{\partial\vartheta} \\ v_\vartheta &= -\dfrac{1}{r\sin\vartheta}\dfrac{\partial\Psi}{\partial r} \end{cases}$$

4.6 Viscous potential flows

if

$$\operatorname{curl}\vec{v} = 0 \quad\rightarrow\quad \vec{v} = \nabla\Phi$$

with the velocity potential

$$\Phi(\vec{r},t)$$

$\vec{v}$ is called a "potential flow". For incompressible fluids we have

$$\operatorname{div}\vec{v} = \operatorname{div}(\nabla\Phi) = \Delta\Phi = 0$$

Navier-Stokes eq. for a potential incompressible flow:

$$\underbrace{\partial_t \nabla\Phi + \frac{1}{2}\nabla(\nabla\Phi)^2 = -\text{grad}\,\frac{p}{\rho_0} + \frac{1}{\rho_0}\vec{f}}_{\text{Euler-equations}} + \underbrace{\nu\,\Delta\,\nabla\Phi}_{\text{friction}} \tag{4.29}$$

but:

$$\Delta\,\nabla\Phi = \nabla\,\underbrace{\Delta\Phi}_{=0} = 0\,!$$

Friction has <u>no</u> influence on the motion of a potential flow.

or

A potential flow moves without friction.

Problem: What to do with the boundary conditions?

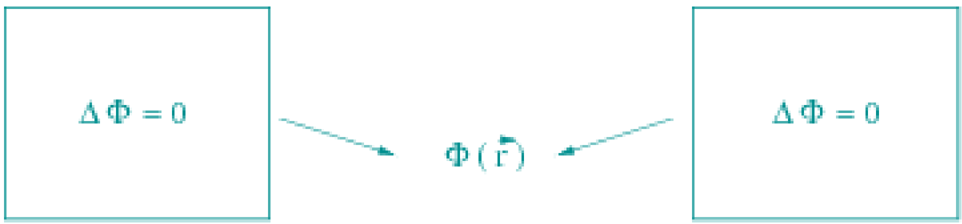
4.7 Boundary layers

Assume a vortex free, potential flow (incompressible)

$$\vec{v} = \nabla\Phi, \qquad \Delta\Phi = 0$$

Now we need boundary conditions for Φ to solve the Laplace equation. There are two possibilities:

1st boundary value problem (Dirichlet) <u>or</u> 2nd boundary value problem (Neumann)



$\Phi\,|_\Omega$ given $\qquad\qquad$ $\frac{\partial\Phi}{\partial\hat{n}}|_\Omega$ given

Now consider a viscous fluid on an impermeable no-slip wall (at Ω):

$$v_{\vec{n}}|_\Omega = \partial_{\vec{n}}\Phi|_\Omega = 0, \qquad \rightsquigarrow \qquad \partial_{\vec{n}}\Phi|_\Omega = 0$$

and

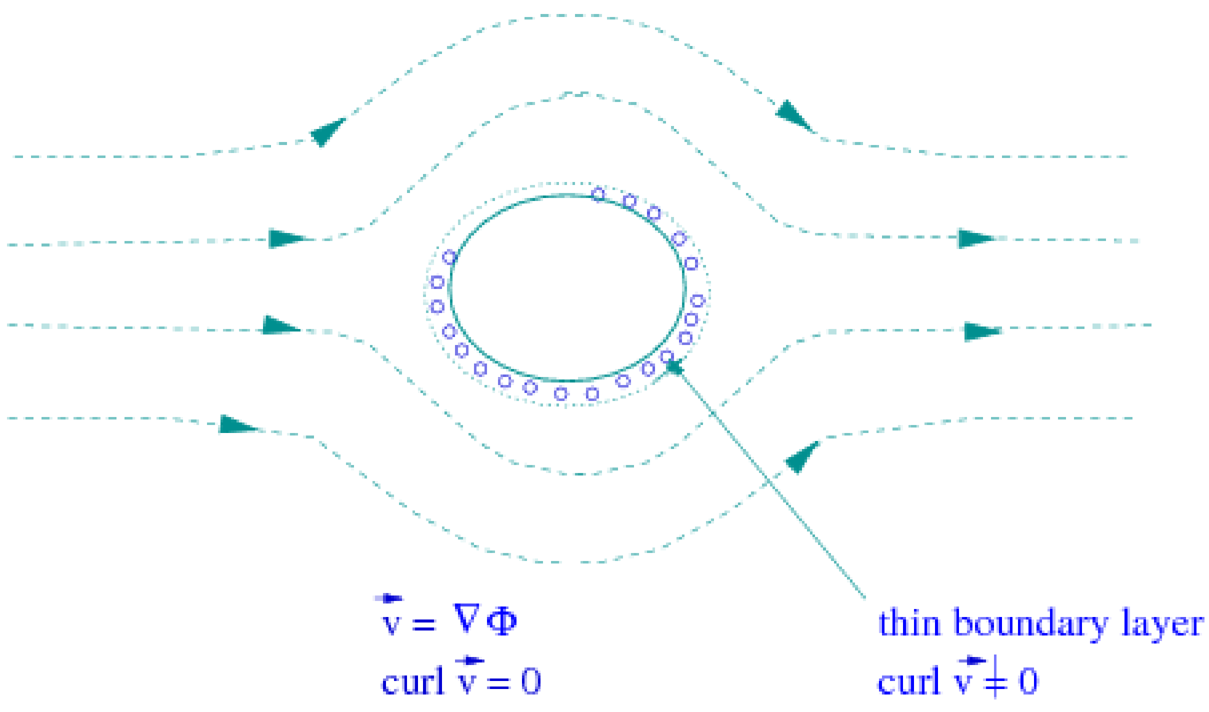
$$v_{\vec{t}}|_\Omega = \partial_{\vec{t}}\Phi|_\Omega = 0, \qquad \rightsquigarrow \qquad \Phi|_\Omega = \text{const}$$

So we have both boundary value problems. But than the system is over determined!

solution: concept of boundary layers

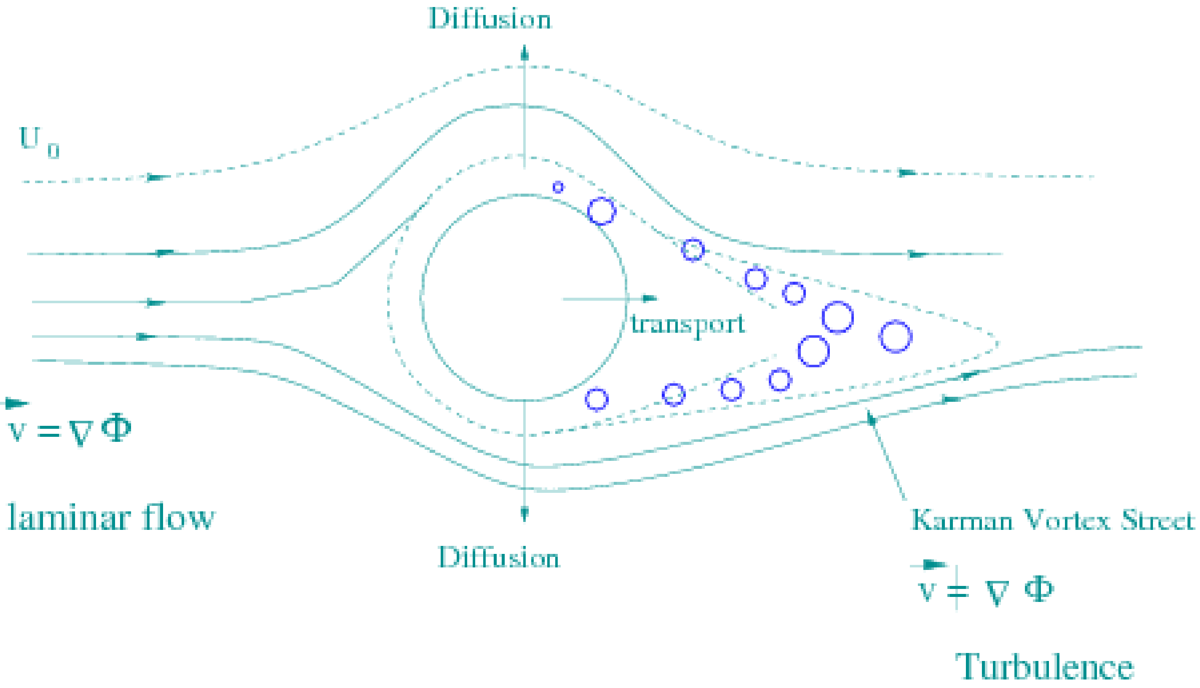
Ludwig Prandtl – Boundary layer theory:

at least close to a wall (or an obstacle) the fluid cannot be potential → vortices and shear flow occur there.



one can derive an equation for the vorticity $\vec{\omega} = \frac{1}{2}\text{curl}\,\vec{v}$ of the form

$$\underbrace{\frac{D\vec{\omega}}{Dt}}_{\text{transport by flow}} = (\vec{\omega}\cdot\nabla)\,\vec{v} + \underbrace{\nu\,\Delta\vec{\omega}}_{\text{diffusion}} \tag{4.30}$$



$R_e \sim U_0$, Reynolds number

R_e characterizes the flow

$R_e < R_e^{crit}$: laminar, boundary layer located at obstacle

$R_e > R_e^{crit}$: turbulent, separation of boundary layer