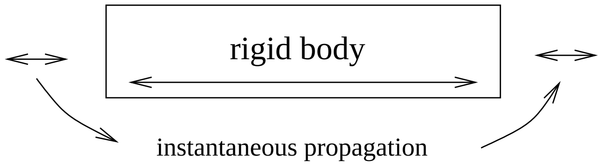
# Chapter 5

# Sound Waves

## 5.1 Preliminaries

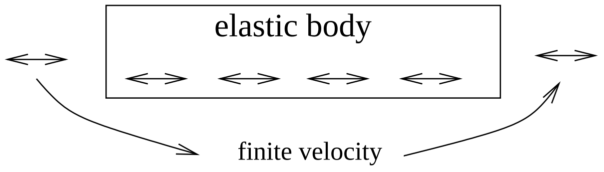
Sound waves exist in solids, liquids and gases. To allow for propagation of sound waves, the medium must be compressible.

An incompressible fluid behaves like a rigid body. The body moves without deformation and oscillations on one side are transmitted instantaneously to the other (arbitrarily distant) side:




this corresponds to an infinite sound speed!

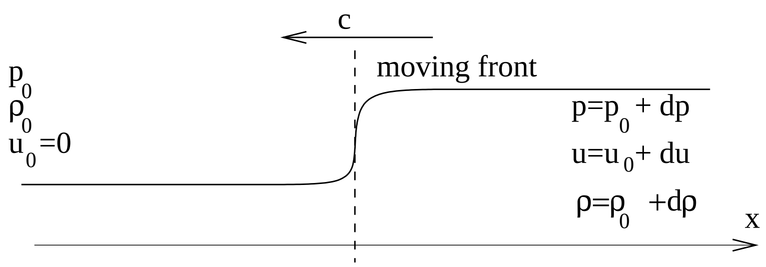
A compressible fluid behaves like an elastic solid. Now oscillations propagate through the solid in form of compression waves. Their speed is finite and depends on the elastic properties, pressure, temperature etc.



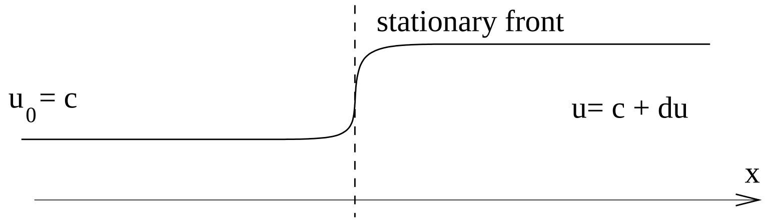
If the amplitude of these compression waves is (infinitesimally) small, they are called "acoustic waves" or "sound waves".

## 5.2 Sound speed

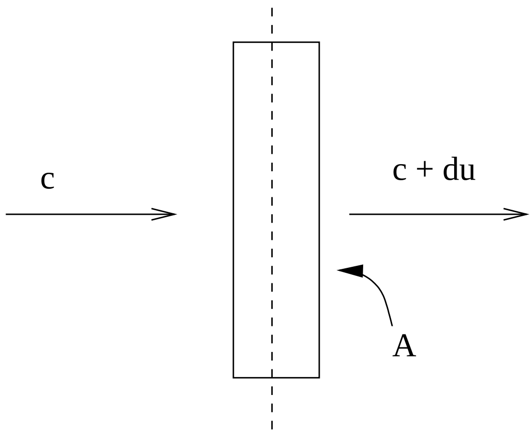
We consider a front that separates two regions with different pressure and density. The front moves to the left with constant velocity $c$. The fluid left from the front is in rest.




In the co-moving frame (with $c$ to the left) one has a stationary front.




Now we take a finite volume around the front with surface $A$



Conservation of mass yields

$$c\,\rho_0\,A = (c+du)\,(\rho_0+d\rho)\,A \approx c\,\rho_0\,A + \rho_0\,A\,du + c\,A\,d\rho$$

or

$$du = -\frac{c}{\rho_0}\,d\rho \tag{5.1}$$

If $d\rho > 0$ (compression wave), the fluid behind the front moves with $du < 0$ (in the direction of the front motion).

Now we take the conservation of the momentum (no friction, Euler eq.):

$$\underbrace{p_0\,A - (p_0+dp)\,A}_{\text{net force on } V} = \underbrace{A\,\rho_0\,c\cdot(c+du) - A\,\rho_0\,c\cdot c}_{\text{change of x-momentum}}$$

or

$$dp = -\rho_0\,c\,du \tag{5.2}$$

Eliminating $du$ from (5.1) and (5.2) one finds the important result

$$\boxed{c^2 = \frac{dp}{d\rho}}$$

To compute the speed of sound one needs a state equation $p = p(\rho)$.

## 5.3 Wave equation for sound waves

For small amplitude waves, viscosity and nonlinearities can be neglected. The linearized Euler eq. and continuity eq. read

$$\rho\,\frac{\partial\vec{v}}{\partial t} = -\mathrm{grad}\,p \tag{5.3}$$

$$\frac{\partial\rho}{\partial t} = -\mathrm{div}\,(\vec{v}\,\rho) \tag{5.4}$$

### 5.3.1 Compression waves

We use the decomposition

$$\rho\,\vec{v} = \rho\,\vec{v}_1 + \rho\,\vec{v}_2$$

with

$$\operatorname{curl}\rho\,\vec{v}_1 = 0, \quad \rightsquigarrow \quad \rho\,\vec{v}_1 = \operatorname{grad}\Phi$$

and

$$\operatorname{div}\rho\,\vec{v}_2 = 0, \quad \rightsquigarrow \quad \rho\,\vec{v}_2 = \operatorname{curl}\vec{A}$$

The first part describes a pure compression without shearing or vortices. The second part corresponds to a shearing without volume change.

Inserting this into (5.3) yields

$$\operatorname{grad}\dot{\Phi} + \operatorname{curl}\dot{\vec{A}} = -\operatorname{grad}p \tag{5.5}$$

and

$$\operatorname{curl}\dot{\vec{A}} = 0$$

The vortices remain constant and are conserved. Then we can integrate (5.5) to

$$\dot{\Phi} = p_0 - p \tag{5.6}$$

with a certain constant $p_0$. Inserting the decomposition into (5.4) one gets

$$\dot{\rho} = -\operatorname{div}\operatorname{grad}\Phi - \underbrace{\operatorname{div}\operatorname{curl}}_{=0}\vec{A} = -\Delta\Phi \tag{5.7}$$

Again we need a state equation of the form $p = p(\rho)$. Then we can differentiate (5.6) with respect to time and use (5.7)

$$\ddot{\Phi} = -\dot{p} = -\frac{dp}{d\rho}\,\dot{\rho} = \frac{dp}{d\rho}\,\Delta\Phi$$

or

$$\boxed{\ddot{\Phi} - c^2\,\Delta\Phi = 0}$$

This is a wave equation for sound waves with phase speed $c^2 = dp/d\rho$. From (5.6) the pressure waves can be computed.

### 5.3.2 State equation

To evaluate $c$, Isaac Newton used the state equation of a perfect gas:

$$p = \rho\, R\, T \tag{5.8}$$

For air at room temperature this gives $c \approx 290$ m/s, compared to $c = 340$ m/s from the experiment. Newton assumed "unclean air" being the reason for the large discrepancy.

About 100 years later, Laplace showed that the compression is not isothermal but adiabatic (or isentrop). The temperature changes during compression periodically. But the motion is so fast, that the temperature fluctuations are not transported to the environment by heat flux.

For an adiabatic process, the relation between pressure and density reads

$$p = \text{const} \cdot \rho^{\gamma}$$

where $\gamma = c_p/c_V$ is the adiabatic exponent and $c_p$, $c_V$ is the specific heat under constant pressure and volume, respectively. For a mono-atomic perfect gas one has $\gamma = 5/3$, for a di-atomic gas $\gamma = 7/5$.

Thus

$$\frac{dp}{d\rho} = \gamma \cdot \text{const} \cdot \rho^{\gamma-1}$$

Using (5.8) one determines the constant to $RT\rho^{1-\gamma}$ and finally finds

$$c = \sqrt{\gamma R\, T}\ ,$$

a value, which is in excellent agreement with the experiment.

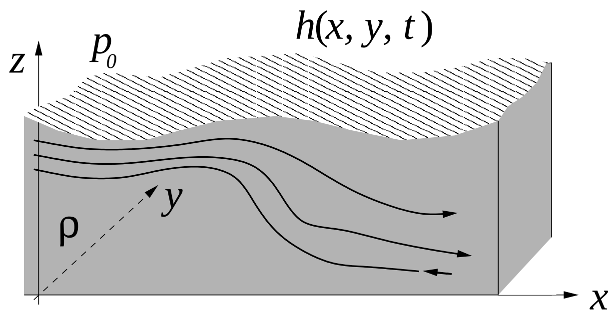
# Chapter 6

# Surface Waves

## 6.1 Preliminaries

We consider waves on the surface of a liquid layer (river, lake, ocean)

$P_0$: external pressure
ρ: density of fluid




- find equation for $h(x,y,t)$
- find the internal motion of the fluid $\vec{v}$
- can instabilities occur ? $\rightarrow$ (Part III)

Assumptions and approximations:

- viscosity is not important $\rightarrow$ Euler equations
- no vorticity, $\text{curl}\,\vec{v} = 0 \quad \rightarrow$ Potential flow
- incompressible fluid, $\text{div}\,\vec{v} = 0$

- small surface deflection, $\frac{\mid h-h_0 \mid}{h_0} \ll 1$

## 6.2 Gravity waves

### 6.2.1 equations for flow

Euler equations ($\rho = \text{const}$)

$$\dot{\vec{v}} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\text{grad}\,(P+U)$$

with (Potential flow)

$$\vec{v} = \nabla\Phi$$

and the formula

$$(\vec{v}\cdot\nabla)\vec{v} = \frac{1}{2}\underbrace{\nabla v^2}_{\nabla(\nabla\Phi)^2} - \vec{v}\times\underbrace{(\nabla\times\vec{v})}_{=0}$$

We find

$$\nabla\left[\dot{\Phi} + \frac{1}{2}(\nabla\Phi)^2\right] = -\nabla\left[\frac{P+U}{\rho}\right] \tag{6.1}$$

or, after integration

$$\begin{aligned} \dot{\Phi} &= -\frac{P+U}{\rho} - \frac{(\nabla\Phi)^2}{2} \\ \Delta\Phi &= 0 \end{aligned} \tag{6.2}$$

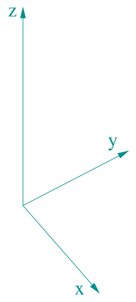
This are the basic equation for an incompressible, vortex free fluid (cmp. part I, chapt. 3.4)

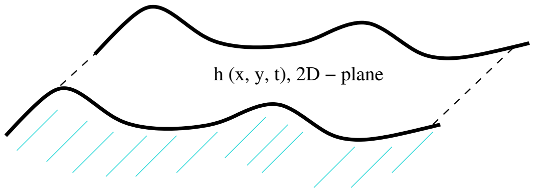
The water in a constant gravitation field has the potential energy:

$$U = \rho g z + U_0$$

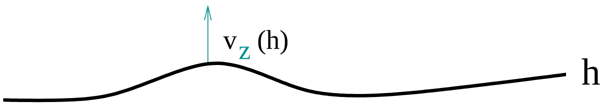
### 6.2.2 Equation for the location of the free surface

Let the surface be located at $z = h(x,y,t)$




If there is a vertical velocity component, the surface moves with that velocity: $\dot{h} = v_z(z = h)$




but also a horizontal velocity takes the surface with it, according to:

$$\dot{h} = -v_x(z = h)\partial_x h$$


both together yields (in three dimensions)

$$\dot{h} = -\underbrace{v_x|_h}_{\partial_x\Phi}\partial_x h - \underbrace{v_y|_h}_{\partial_y\Phi}\partial_y h + \underbrace{v_z|_h}_{\partial_z\Phi} \tag{6.3}$$

or, using the potential:

$$\dot{h} = -\nabla_2\Phi|_h \cdot \nabla_2 h + \partial_z\Phi|_h \tag{6.4}$$

Now we evaluate eq. (6.2) at the surface $z = h$:

$$\dot{\Phi}|_h = -\underbrace{g(h-h_0)}_{U(z=h)} - \frac{1}{2}(\nabla\Phi)_h^2 \tag{6.5}$$

where we used

$$U = \rho g z + U_0, \qquad \text{with} \quad U_0 = P_0 - \rho g h_0$$

### 6.2.3 Basic equations and linear solutions

$$\begin{aligned}
\Delta\Phi &= 0 && (6.6)\\
\dot{\Phi}_h &= -g(h-h_0) - \frac{1}{2}(\nabla\Phi)_h^2 && (6.7)\\
\dot{h} &= -\nabla\Phi|_h \cdot \nabla h + \partial_z\Phi|_h && (6.8)
\end{aligned}$$

Now we assume that the basic state is that of a flat surface $h = h_0$ where the fluid is in rest, $\Phi = 0$ (hydrostatic solution)

Consider small deviations from that state

$$\eta(x,t) = h(x,t) - h_0, \qquad \Phi(x,t) \tag{6.9}$$

eqs. (6.7), (6.8) can be linearized:

$$\left.\begin{aligned} \dot{\Phi}|_z &= -g\eta \\ \dot{\eta} &= \partial_z\Phi|_h \end{aligned}\right\} \quad \Longrightarrow \quad \partial_z\Phi + \frac{1}{g}\ddot{\Phi} = 0 \tag{6.10}$$

We assume a solution in form of waves:

$$\Phi = \xi(t)\cdot f(z)\,\mathrm{e}^{ikx} \tag{6.11}$$

inserting this into (6.6) yields

$$f'' - k^2 f = 0 \qquad \rightarrow \qquad f(z) \sim \ \mathrm{e}^{\pm|k|(z-h_0)}$$

and with the boundary condition (infinitely deep layer)

$$\Phi(z\to-\infty) = 0 \qquad \to f(z) \sim e^{|k|(z-h_0)} \tag{6.12}$$

Substitute (6.11), (6.12) into (6.10) gives

$$|k|\zeta + \frac{1}{g}\ddot{\zeta} = 0 \qquad \rightarrow \qquad \zeta(t) = e^{\pm i\omega t}\ , \tag{6.13}$$

the equation of an harmonic oscillator with the frequency

$$\omega = \sqrt{|k|g}$$

Thus we have as a solution of the linearized problem

$$\Phi = Ae^{|k|(z-h_0)}\cos(kx \pm \omega t) \tag{6.14}$$
$$h = h_0 \pm A\sqrt{\frac{k}{g}}\sin(kx \pm \omega t) \tag{6.15}$$

and from there the velocity components

$$v_x = \partial_x \Phi = -kAe^{|k|(z-h_0)}\sin(kx \pm \omega t) \tag{6.16}$$
$$v_z = \partial_z \Phi = |k|Ae^{|k|(z-h_0)}\cos(kx \pm \omega t) \tag{6.17}$$

This corresponds to traveling waves with the phase velocity

$$c = \frac{\omega}{k}$$

The dispersion relation reads

$$\omega = \sqrt{kg}$$

Using this, the phase velocity can be expressed as

$$c = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$$

– the longer the wave length, the faster the wave propagates

How do the trajectories of volume element (its path) look?

To answer this, one has to solve the system

$$\frac{dx}{dt} = v_x = -kAe^{|k|(z-h_0)}\sin(kx \pm \omega t) \tag{6.18}$$
$$\frac{dz}{dt} = v_z = |k|Ae^{|k|(z-h_0)}\cos(kx \pm \omega t) \tag{6.19}$$

two coupled nonlinear ODE’s which can be solved only numerically.

Approximation: $|\vec{v}| \ll c$

With the initial condition $x_0 = x(t=0), z_0 = z(t=0)$ one can integrate

$$x(t) = x_0 + \int_0^t v_x(x_0, z_0, t')dt' = x_0 + a(\cos(kx_0 - \omega t) - \cos kx_0) \quad (6.20)$$

$$z(t) = z_0 + \int_0^t v_z(x_0, z_0, t')dt' = z_0 - b(\sin(kx_0 - \omega t) - \sin kx_0) \quad (6.21)$$

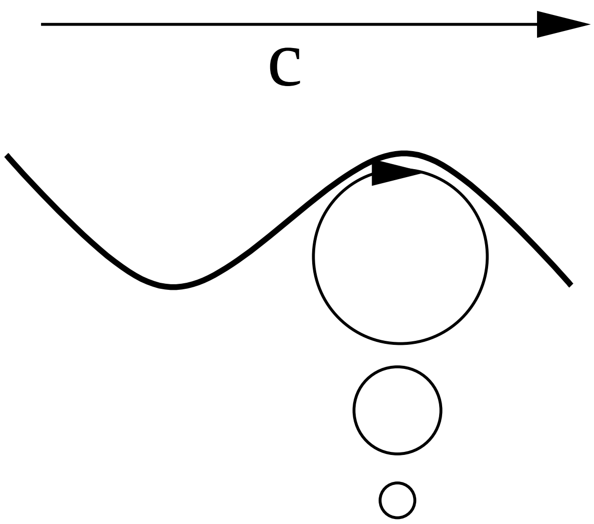
with

$$a = -\frac{Ak}{\omega} e^{|k|(z_0 - h_0)} \quad (6.22)$$

$$b = \frac{A|k|}{\omega} e^{|k|(z_0 - h_0)} \quad (6.23)$$

– volume elements travel on circles with radius $|a| = |b| \sim e^{|k|z_0}$

– in time average, particles don't travel at all!



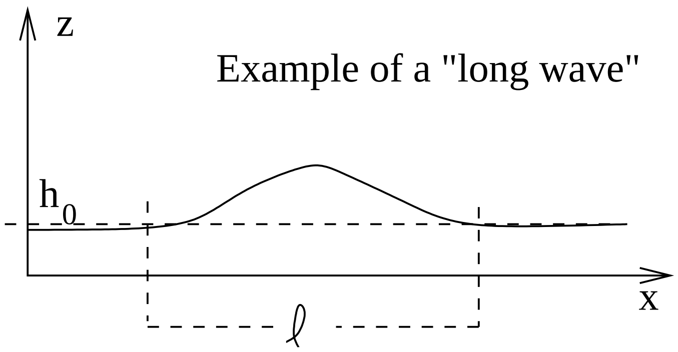

But: nonlinear corrections leads to the so-called "Stokes drift", an average velocity

$$|\vec{v}_s| \sim a^2$$

and parallel to $\vec{k}$.

## 6.3 The Shallow Water equations

Now we consider surface deformations in form of long waves. This does not mean only harmonic waves but can be any other form. It is important that the dimension (extension) in horizontal direction is large compared to the depth of the fluid.




$$\delta = \frac{h_0}{\ell} \ll 1$$

Examples for “long waves” are:

- ocean waves near the shore
- Tsunamis
- Waves on a canal

To arrive at a dimensionless formulation of the problem, the variables of eqs. (6.6), (6.7), (6.8) are scaled in the following way:

$$x = \tilde{x} \cdot \ell, \qquad z = \tilde{z} \cdot h_0, \qquad h = \tilde{h} \cdot h_0 \tag{6.24}$$

$$t = \tilde{t} \cdot \tau \qquad \Phi = \tilde{\Phi} \cdot \frac{\ell^2}{\tau} \tag{6.25}$$

Then eqs. (6.6), (6.7), (6.8) read

$$0 = \partial^2_{\tilde{z}\tilde{z}} \tilde{\Phi} + \delta^2 \partial^2_{\tilde{x}\tilde{x}} \tilde{\Phi} \tag{6.26}$$

$$\dot{\tilde{\Phi}} = -G\,(\tilde{h} - 1) - \frac{1}{2} (\partial_{\tilde{x}} \tilde{\Phi})^2 - \frac{1}{2\delta^2} (\partial_{\tilde{z}} \tilde{\Phi})^2 \tag{6.27}$$

$$\delta^2 \dot{\tilde{h}} = -\delta^2 \partial_{\tilde{x}} \tilde{\Phi} \cdot \partial_{\tilde{x}} \tilde{h} + \partial_{\tilde{z}} \tilde{\Phi} \tag{6.28}$$

(from here, we suppress the tildes). The non-dimensional number

$$G = \frac{g \cdot h_0 \cdot \tau^2}{\ell^2}$$

is called “gravitation number”.

Trick: we solve (6.26) by iteration (systematic perturbation analysis with respect to small $\delta$):

$$\Phi = \Phi^{(0)} + \delta^2 \Phi^{(2)} + \delta^4 \Phi^{(4)} + \dots \tag{6.29}$$

this inserted into (6.26) gives:

$$\partial^2_{zz}\Phi^{(0)} + \delta^2\left(\partial^2_{zz}\Phi^{(2)} + \partial^2_{xx}\Phi^{(0)}\right) + \delta^4\left(\partial^3_{zz}\Phi^{(4)} + \partial^2_{xx}\Phi^{(2)}\right) + \dots = 0 \tag{6.30}$$

Since $\delta$ can be arbitrary, terms with the same order of $\delta$ must vanish:

- order $\delta^0$

$$\partial^2_{zz}\Phi^{(0)} = 0 \qquad \rightarrow \qquad \Phi^{(0)} = f_1(x,t) + f_2(x,t)\cdot z \tag{6.31}$$

  with the boundary condition on the ground ($z = 0$)

$$v_z(z=0) = \left.\frac{\partial \Phi}{\partial z}\right|_{z=0} = f_2 = 0 \quad \rightsquigarrow \quad f_2 = 0 \tag{6.32}$$

  and finally the important result

$$\Phi^{(0)} = \Phi^{(0)}(x,t) \tag{6.33}$$

- order $\delta^2$:

$$\begin{aligned} \partial^2_{zz}\Phi^{(2)} &= -\partial^2_{xx}\Phi^{(0)} = -\Phi^{(0)''} && (6.34)\\ \rightarrow \quad \Phi^{(2)}(x,z,t) &= -\Phi^{(0)''}\cdot\frac{z^2}{2} + \underbrace{f_3(x,t)}_{=0 \text{ (b. c)}}\cdot z + f_4(x,t) && (6.35)\end{aligned}$$

- order $\delta^4$
  - in the same way …

We write down the result up to the order $\delta^4$:

$$\Phi(x,z,t) = \Phi^{(0)}(x,t) + \delta^2 \left[ -\Phi^{(0)''} \cdot \frac{z^2}{2} + f_4(x,t) \right] + \delta^4 \left[ \Phi^{(0)''''} \cdot \frac{z^4}{24} - f_4'' \cdot \frac{z^2}{2} + f_6(x,t) \right] + \ldots \quad (6.36)$$

– we know the z-dependence of $\Phi$ explicitly !!

– if $\Phi^{(0)}(x,t)$ is known, $\Phi(x,z,t)$ can be determined.

Now we insert this into (6.27), (6.28) and take the lowest, non-trivial order:

$$\dot{\Phi}^{(0)} = -G(h-1) - \frac{1}{2}\left(\partial_x \Phi^{(0)}\right)^2 \quad (6.37)$$

$$\dot{h} = -\partial_x \Phi^{(0)} \cdot \partial_x h - h \cdot \partial_{xx}^2 \Phi^{(0)} \quad (6.38)$$

or in two horizontal dimensions $(x,y)$

$$\dot{\Phi} = -G(h-1) - \frac{1}{2}(\nabla_2 \Phi)^2 \quad (6.39)$$

$$\dot{h} = -\nabla_2 \Phi \cdot \nabla_2 h - h \cdot \Delta_2 \Phi \quad (6.40)$$

These are the Shallow Water equations.

Advantage: only two equations instead of three

Big advantage: one spatial dimension is eliminated!

$$3D \rightarrow 2D$$
$$2D \rightarrow 1D$$

### 6.3.1 The linearized Shallow Water equations

We consider small deviations $\eta$ from the constant depth $h_0 = 1$:

$$\eta = h - 1$$

Then $\Phi$ is also small and (6.37), (6.38) or (6.39), (6.40) can be linearized:

$$\dot{\Phi} = -G\eta \quad (6.41)$$
$$\dot{\eta} = -\Delta_2\Phi \quad (6.42)$$

Differentiating (6.42) with respect to time and eliminating $\dot{\Phi}$ yields a wave equation for $\eta$

$$\boxed{\ddot{\eta} - c^2\Delta_2\eta = 0}$$

with the phase velocity

$$c = \sqrt{G}$$

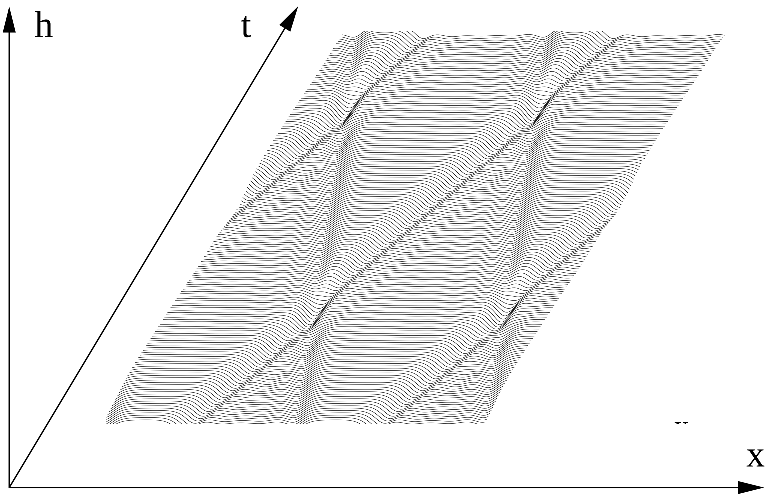
rescaling all variables gives the velocity in dimensional form

$$c = \sqrt{gh_0}$$

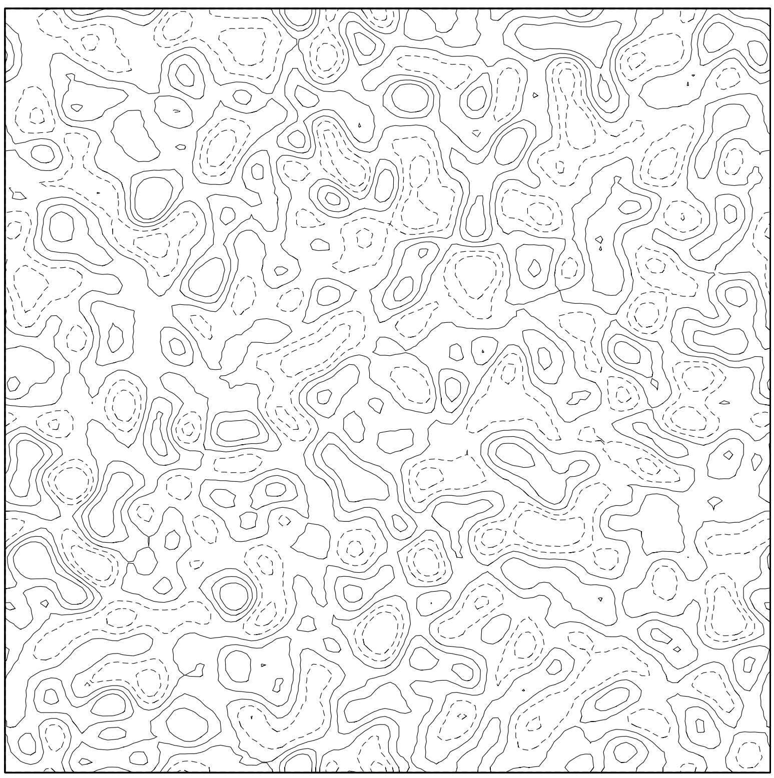
- phase velocity of long waves is constant!
- it depends only on the depth of the layer

### 6.3.2 Numerical solutions of the nonlinear Shallow Water equations

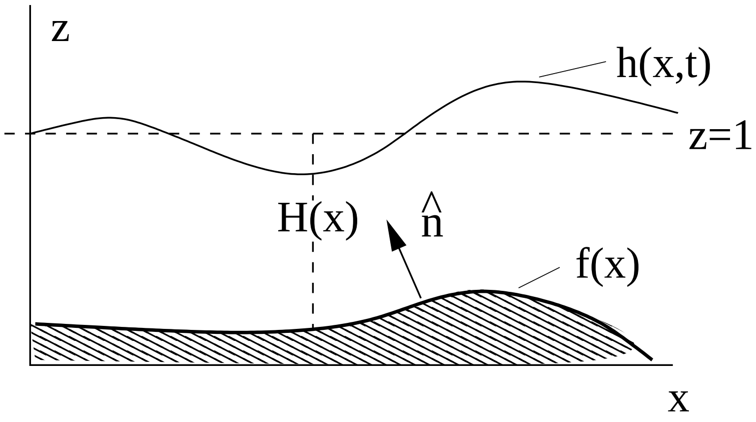
time evolution in 1D




snapshot in 2D



### 6.3.3 Shallow water waves on a modulated ground




boundary condition on the ground:

$$\hat{n} \cdot \vec{v} = \hat{n} \cdot \nabla\Phi = 0$$

Deriving the Shallow Water equations in the same manner as above, this gives rise to two new terms (underlined)

$$\dot{\Phi} = -G(h-1) - \frac{1}{2}(\nabla_2\Phi)^2 \tag{6.43}$$

$$\dot{h} = -(\nabla_2 h)\cdot(\nabla_2\Phi) - h\,\Delta_2\Phi \quad + \quad \underline{(\nabla_2 f)\cdot(\nabla_2\Phi) + f\,\Delta_2\Phi} \tag{6.44}$$

linearizing again yields a wave equation, now of the form

$$\ddot{h} - G\,\nabla_2[H(x,y)\nabla_2 h] = 0 \tag{6.45}$$

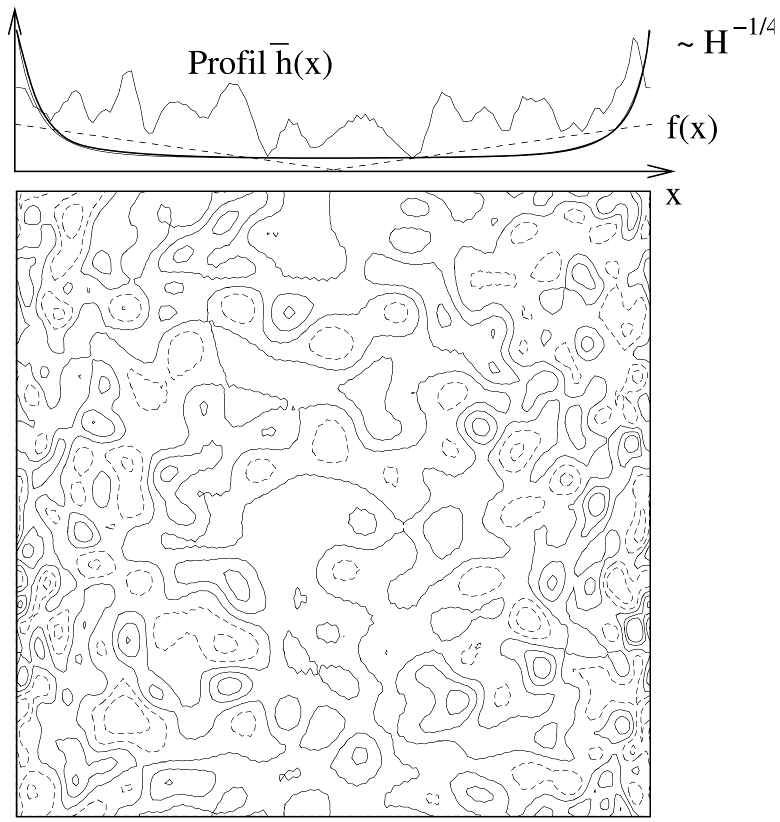
with $H = 1 - f$ denoting the real depth of the flat water. If we neglect $\nabla_2 H$ (corresponding to small changes of the surface on the length scale of the waves), (6.45) describes waves with space dependent velocity

$$c_p(x,y) = \sqrt{G\,H(x,y)} \tag{6.46}$$

It is obvious that waves slow down if they reach a shallower region. In the mean time their wave length decreases:

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega}c_p = \frac{2\pi}{\omega}\sqrt{GH} \quad \sim H^{\frac{1}{2}} \tag{6.47}$$

Numerical solution of waves on a beach with constant slope

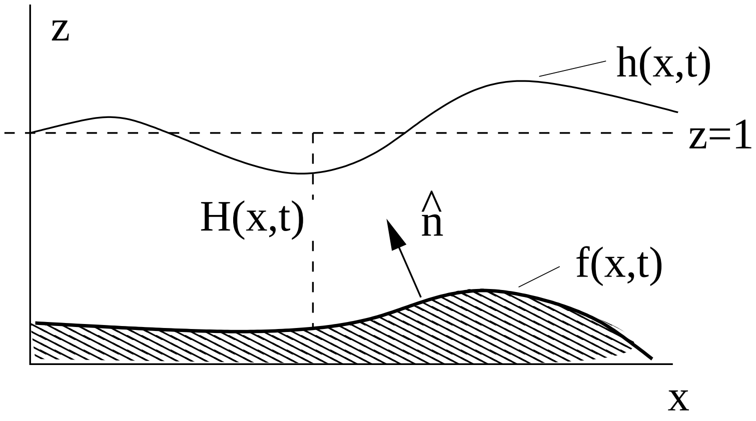



If the ground has a small slope, Green's law can be derived from weakly non-linear theory (see textbooks, e.g. Lamb, *Hydrodynamics*, Cambridge Univ. Press):

$$\boxed{A \sim H^{-\frac{1}{4}}} \tag{6.48}$$

From there one sees that the amplitude of waves increases if they approach the shore. We shall return to this issue in the sect. on Tsunamis.

### 6.3.4 Generation of waves by a time-dependent ground




The boundary conditions now have the form:

$$\hat{n} \cdot \vec{v} = \hat{n} \cdot \nabla\Phi = \dot{f}$$

This gives another term (underlined)

$$\dot{\Phi} = -G(h-1) - \frac{1}{2}(\nabla_2\Phi)^2 \qquad (6.49)$$

$$\dot{h} = -(\nabla_2 h)\cdot(\nabla_2\Phi) - h\,\Delta_2\Phi + (\nabla_2 f)\cdot(\nabla_2\Phi) + f\,\Delta_2\Phi \quad + \quad \underline{\dot{f}} \qquad (6.50)$$

If $\Phi = \text{const}$ (fluid in rest) $\quad\rightarrow\quad \dot{h} = \dot{f} \quad\rightarrow\quad h(t) = f(t) \quad + \quad \text{const}$

- the ground motion is equal to the surface motion
- no time delay, reason: fluid is assumed to be incompressible
- A ground motion may generate waves:
  linearized wave equation with $H = 1 - f$:

$$\frac{1}{G}\ddot{\Phi} - \nabla_2[H\nabla_2\Phi] = \dot{f} = -\dot{H} \qquad (6.51)$$

This is an inhomogeneous wave equation. It can be formally solved using an appropriate Green's function:

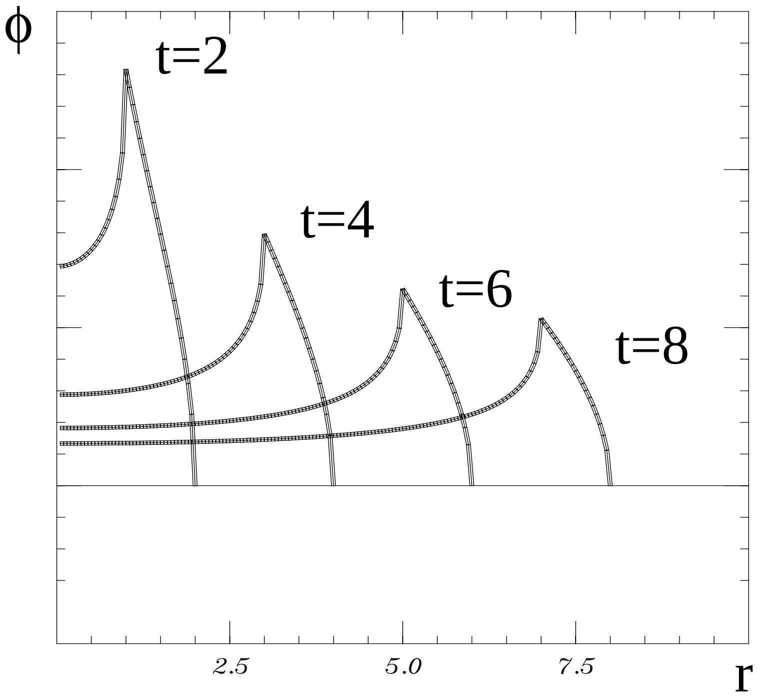
$$\Phi(x,y,t) = \iint dx'dy' \int dt'\, D(x-x', y-y', t-t')\, \dot{H}(x',y',t') \qquad (6.52)$$

Example

consider the following localized ground motion:

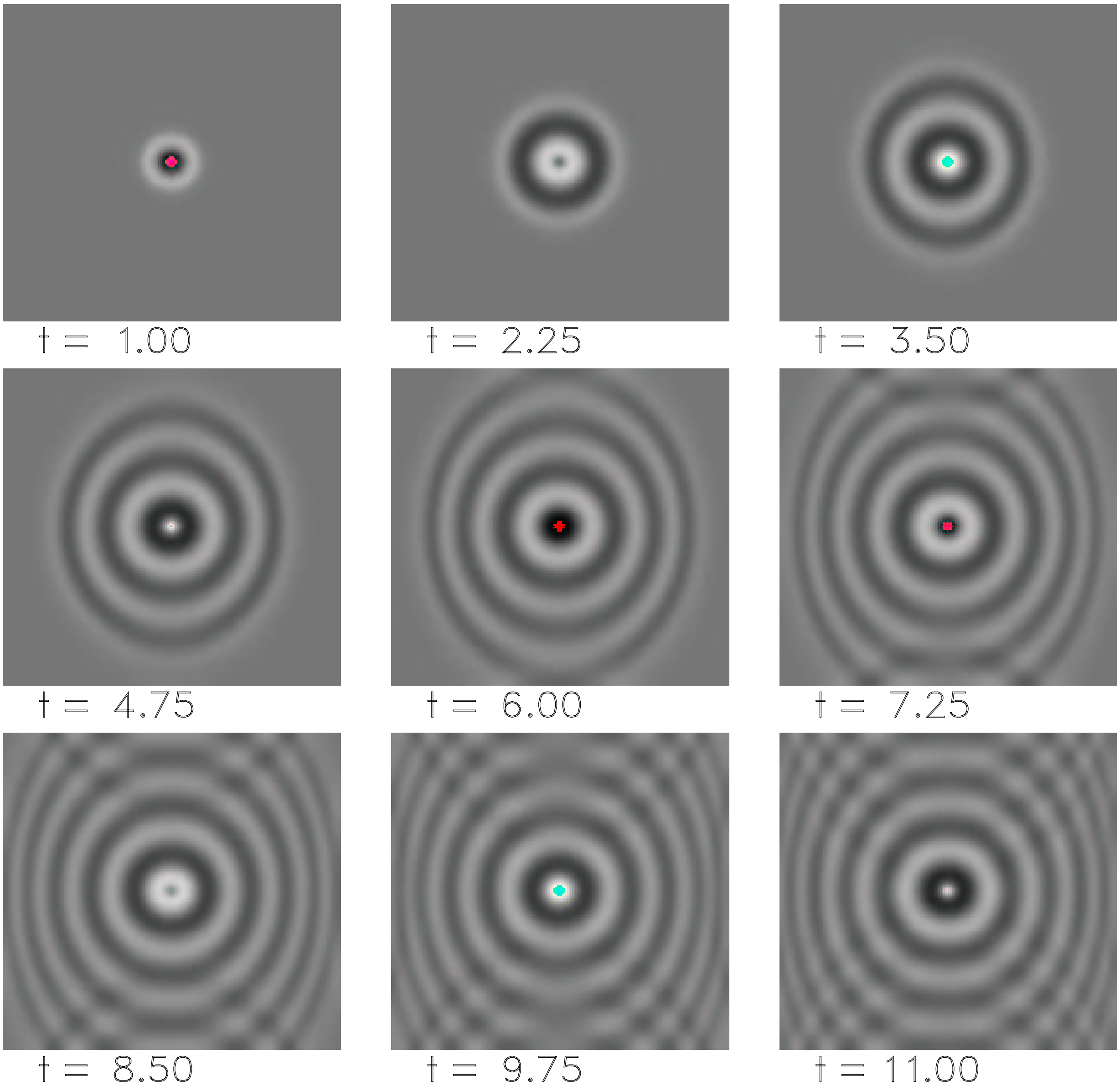
$$H(x,y,t) = \begin{cases} 1 + v_0 \cdot t\ \delta(x)\ \delta(y) & , \quad 0 \le t \le t_0 \\ 1 + v_0 \cdot t_0\ \delta(x)\ \delta(y) & , \quad t_0 < t \end{cases} \tag{6.53}$$


solution: circular waves




snapshots at various times $t$

# Numerical solution in two dimensions




We chose

$$f(x,y,t) = a\, e^{(-r^2/\beta^2)} \cos \Omega t \tag{6.54}$$

- oscillating ground localized at $r = 0$ with a gaussian distribution.
- slopy ground (ramps in $x$-direction, minimum in the center).

### 6.3.5 Tsunamis

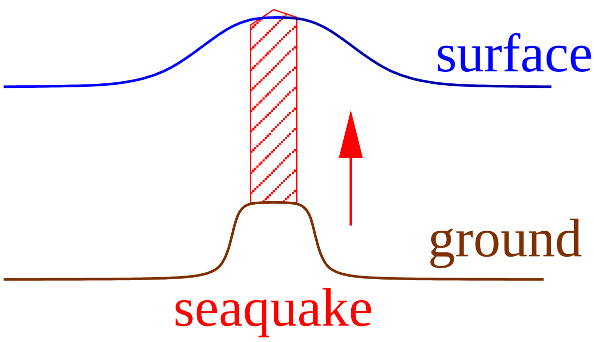
The notion “Tsunami” was coined by Japanese fishermen and means “wave in harbor”. The fishermen went out to the sea during the night for fishing. On their return, they found the harbor destroyed by a flood. Since they didn’t notice anything unusual on the open sea, they thought that these waves were generated in the harbor.

- Tsunamis are caused by seaquakes or landslides
- More than 80 Tsunamis observed in the last 10 years
  - Christmas 2004, Sri Lanka, India, Thailand, more than 200 000 victims
  - Lissabon 1755, caused by the big earthquake 60 000 victims
  - Krakatau 1883, a wave was generated that traveled 7 times round the earth
  - Japan, 1896, a Tsunami called “Sanriku” caused waves with amplitudes up to 23 meters

What is the difference between a Tsunami and waves generated by wind?

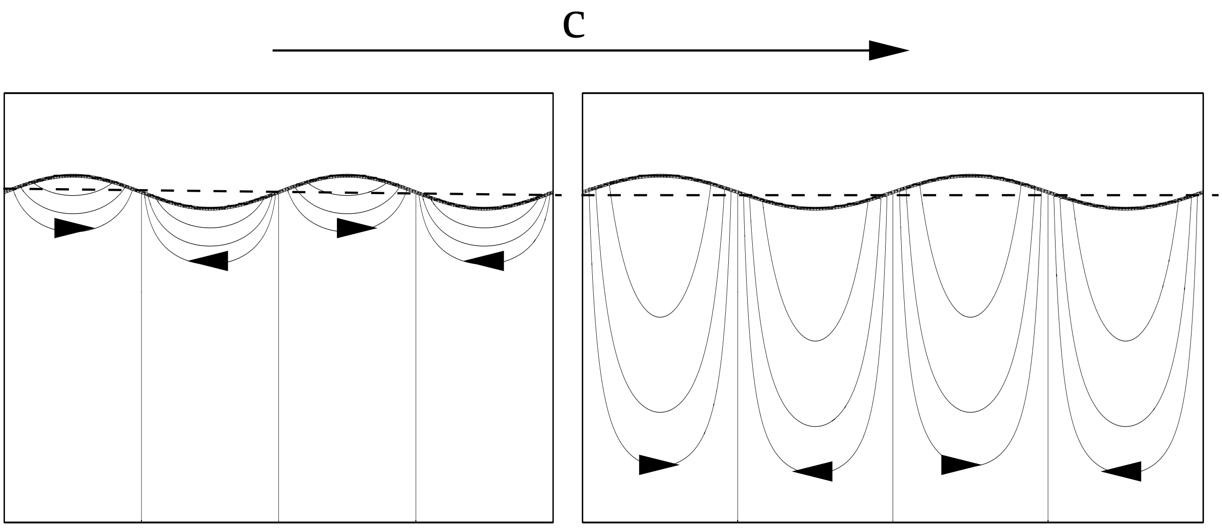
Wind accelerates the fluid on a thin layer at the surface of the sea. Waves generated by wind are short waves or *deep water waves*.

Due to the generation of a Tsunami on the ground of the sea, the whole water column over the seismic center is elevated:




Thus, the fluid over the whole depth moves. Although the fluid motion is rather slow compared to that caused by wind waves, its kinetic energy is enormous due to the large mass in motion

Waves generated by a seaquake are *long waves* or *shallow water waves*.




deep water waves (short waves) shallow water waves (long waves)

- On the high seas (off shore) the wave amplitude is very small: 10 - 50 cm
- The wave length is $> 100$ km
- The water depth is 4 - 7 km
- For Tsunamis, the Shallow-Water theory applies

There we found a relation between phase velocity and water depth:

$$c = \sqrt{g\,h_0}$$

If we use $g = 9,81$ m/s$^2$ and $h_0 = 4000$ m we find

$$c \quad \approx \quad 200\,\text{m/s} \quad \approx \quad 700\,\text{km/h}$$

- A Tsunami may cross an ocean within a few hours!
- There is almost no damping, because the particle velocity is very slow.

$$v = \frac{A}{h_0}\,c$$

with $A$ = 50 cm, $h_0$ = 4000 m, $c = 200$ m/s one gets

$$v \approx 2.5 \text{ cm/s}$$

This cannot be measured on the surface, because it is completely covered by the natural motion (wind). Tsunamis can only be detected well below the surface, where the water is usually not moving (or only in large scaled streams).

For the frequency, we can estimate

$$\nu = \frac{\omega}{2\pi} = \frac{c \cdot k}{2\pi} = \frac{c}{\lambda}$$

Taking $\lambda = 100$ km and $c = 200$ m/s one has $\nu \approx 0,002$ Hz, corresponding to $\Delta t \approx 500$ s between two consecutive waves.

From Green's law we know that the amplitude increases by approaching the shore:

$$A \sim H^{-\frac{1}{4}}$$

The water velocity is also a function of the depth:

$$v = \frac{A}{H}\, c$$

Since $c \sim H^{\frac{1}{2}}$ we finally have a rather strong increase of the water velocity while a Tsunami reaches the shores:

$$\boxed{\frac{v}{v_0} = \left(\frac{H_0}{H}\right)^{\frac{3}{4}}} \tag{6.55}$$

Taking as an example $H_0 = 5000$ m (high seas) and $v_0 = 10$ cm/s, this yields at the shore ($H = 10$ m) $v \approx 10$ m/s.