Chapter 8

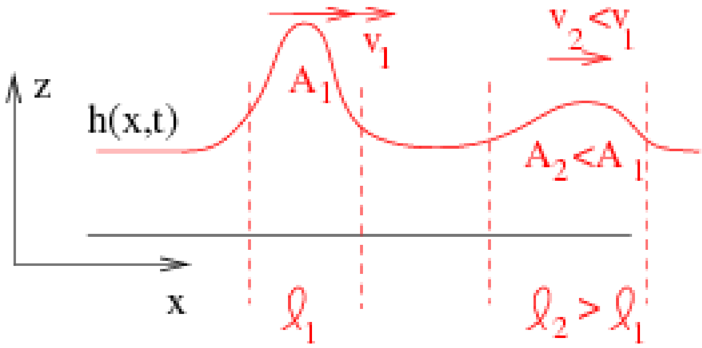
Solitons

8.1 Discovery

- discovered by John Scott Russell in 1834
- localized states of elevated (or depressed) surface
- only in one spatial dimension possible (narrow channels)
- certain specific properties concerning speed, length, interaction

The discovery of John Scott Russell (in his own words)

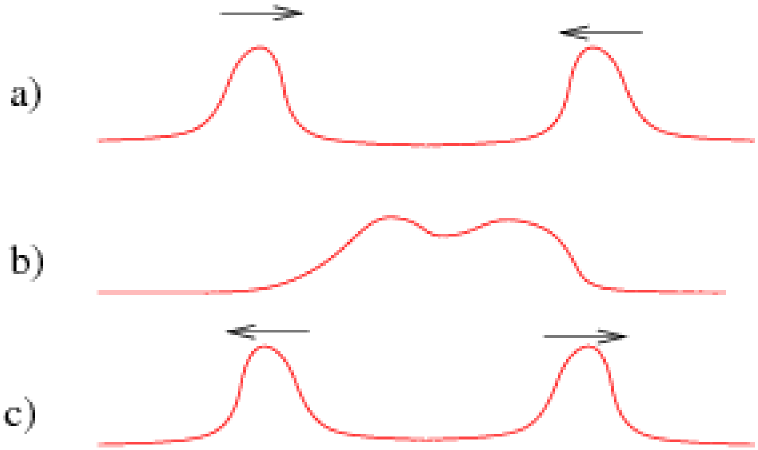
"I believe I shall best introduce the phenomenon by describing the circumstances of my own first acquaintance with it. I was the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot an a half in height. It's height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation, a name which it now very generally bears."



combination of linear and nonlinear behavior

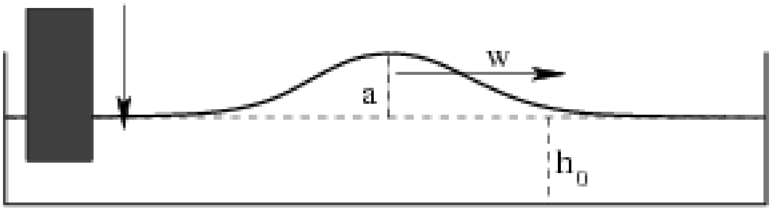
$$A \sim v$$

$$\ell \sim \frac{1}{\sqrt{v}} \sim \frac{1}{\sqrt{A}}$$



The shapes (a) are conserved, even after (c) a collision (b) of two solitons.

Experiments of Russell:



$$w = \sqrt{g(h_0 + a)} \tag{8.1}$$

$a \to 0$: relation for shallow water waves.

but: non-linearities are important (small corrections).

60 years later: Solitons are solutions of the Korteweg-de Vries-equation.

8.2 The Korteweg – de Vries equation

derived by D.J. Korteweg and G. de Vries (1895) from the Euler equations in a systematic way.

The starting point are the shallow water equations:

1st step

we use

$$h = 1 + \alpha\,\eta, \qquad \Phi = \alpha\,\bar{\Phi}$$

α is a small parameter (amplitude), $\alpha \ll 1$

$$\begin{aligned} \dot{\bar{\Phi}} + G\,\eta + \frac{\alpha}{2}\,(\partial_x\bar{\Phi})^2 + \frac{\alpha}{2\delta^2}\,(\partial_z\bar{\Phi})^2 &= 0 \\ \partial_z\bar{\Phi} - \delta^2\dot{\eta} - \delta^2\alpha\,(\partial_x\bar{\Phi})(\partial_x\eta) &= 0 \\ \partial^2_{zz}\bar{\Phi} + \delta^2\partial^2_{xx}\bar{\Phi} &= 0 \end{aligned} \tag{8.2}$$

(in the following we omit the bars).

for $\alpha = 0$, one finds the linear wave equation

$$\partial^2_{xx}\Phi - \frac{1}{G}\,\ddot{\Phi} = 0 \quad \rightarrow \quad \text{waves with } c_0 = \pm\sqrt{G} \tag{8.3}$$

2nd step

assume $\alpha = \delta^2$

3rd step

change to moving frame $\quad x = \tilde{x} - c_0 t$

$$\Phi = \Phi(x - c_0 t, \alpha t), \qquad \eta = \eta(x - c_0 t, \alpha t) \tag{8.4}$$

$$\dot{\Phi} = \alpha\, \partial_t \Phi - c_0 \partial_x \Phi, \qquad \dot{\eta} = \alpha\, \partial_t \eta - c_0 \partial_x \eta \tag{8.5}$$

Inserting this into eqs. (8.2)

$$\alpha\, \dot{\Phi} - c_0 \partial_x \Phi + G\, \eta + \frac{\alpha}{2}\, (\partial_x \Phi)^2 + \frac{1}{2}\, (\partial_z \Phi)^2 = 0 \tag{8.6}$$

$$\partial_z \Phi - \alpha^2 \dot{\eta} + \alpha\, c_0\, \partial_x \eta - \alpha^2 (\partial_x \Phi)(\partial_x \eta) = 0 \tag{8.7}$$

$$\partial^2_{zz} \Phi + \alpha\, \partial^2_{xx} \Phi = 0 \tag{8.8}$$

<u>4th step</u>

expansion with respect to α

$$\eta = \eta^{(0)} + \alpha \eta^{(1)} + \alpha^2 \eta^{(2)} + \quad \dots \tag{8.9}$$

$$\begin{aligned} \Phi(x,z,t) &= \Phi^{(0)}(x,t) + \alpha \left[-\Phi^{(0)''} \cdot \frac{z^2}{2} + f_4(x,t) \right] \\ &+ \alpha^2 \left[\Phi^{(0)''''} \cdot \frac{z^4}{24} - f_4'' \cdot \frac{z^2}{2} + f_6(x,t) \right] + \dots \end{aligned} \tag{8.10}$$

(for the latter expansion see sect. 6.3, eq. (6.36)).

<u>5th step</u>

compare orders of α^n

$\boxed{\alpha^0}$

$$eq.(8.6) \quad \longrightarrow \quad G\, \eta^{(0)} - c_0\, \partial_x \Phi^{(0)} = 0$$

we use $G = c_0^2$:

$$\partial_x \Phi^{(0)} = c_0 \eta^{(0)}$$

$\boxed{\alpha^1}$

$$eq.(8.6) \quad \longrightarrow \quad c_0 \eta^{(1)} - \partial_x f_4 + \frac{1}{c_0} \dot{\Phi}^{(0)} + \frac{1}{2} \partial^3_{xxx} \Phi^{(0)} + \frac{1}{2c_0} (\partial_x \Phi^{(0)})^2 = 0$$

$\boxed{\alpha^2}$

$$eq.(8.7) \quad \longrightarrow \quad \frac{1}{6}\partial^4_{xxxx}\Phi^{(0)} - \partial^2_{xx} f_4 - \partial^2_{xx}\Phi \cdot \eta^{(0)} - \dot{\eta}^{(0)} + c_0\partial_x\eta^{(1)} - (\partial_x\Phi^{(0)})(\partial_x h^{(0)}) = 0$$

<u>6th step</u>

elimination of f_4 and $\Phi^{(0)}$

– differentiate $\boxed{\alpha^1}$ by ∂_x and use $\boxed{\alpha^0}$

$$c_0\partial_x\eta^{(1)} - \partial^2_{xx} f_4 + \dot{\eta}^{(0)} + \frac{1}{2}c_0\partial^3_{xxx}\eta^{(0)} + c_0\eta^{(0)}\partial_x\eta^{(0)} = 0$$

– subtract $\boxed{\alpha^2}$ and use $\boxed{\alpha^0}$

$$2\dot{\eta}^{(0)} + \frac{1}{3}c_0\partial^3_{xxx}\eta^{(0)} + 3c_0\eta^{(0)}\partial_x\eta^{(0)} = 0$$

scaling

$$t = \frac{2}{9c_0}t', \qquad x = \frac{1}{3}x'$$

$$\boxed{\dot{\eta} = -\partial^3_{xxx}\eta - \eta\partial_x\eta}$$

This is the Korteweg-de Vries equation (KdV).

In addition, one needs boundary conditions and initial conditions for η.

Solution of KdV: solitons (and more), see next sects.

8.3 Numerical solutions of the KdV equation

In this sect. we shall introduce the standard numerical method for the KdV.

8.3.1 Conserved quantities

It is important to know that there are certain quantities, which are conserved.

Consider

$$\bar{\eta}(t) = \frac{1}{L}\int_0^L \eta(x,t)\,dx \tag{8.11}$$

the mean height. If the total mass (of an incompressible fluid) is conserved, this must be true also for its volume. Since the volume is proportional to the mean height, we have

$$\bar{\eta} = \text{constant}$$

Of course this is also a feature of the KdV:

$$\begin{aligned}
\dot{\bar{\eta}} &= \frac{\partial}{\partial t}\frac{1}{L}\int_0^L \eta(x,t)dx = \frac{1}{L}\int_0^L \dot{\eta}(x,t)dx \\
&\overset{(kdV)}{=} -\frac{1}{L}\int_0^L \partial^3_{xxx}\eta\; dx - \frac{1}{L}\int_0^L \eta\partial_x\eta\; dx \\
&= -\frac{1}{L}\partial^2_{xx}\eta\Big]_0^L - \frac{1}{2}\eta^2\Big]_0^L = 0
\end{aligned} \tag{8.12}$$

where periodic lateral boundary conditions have been assumed.

Another conserved quantity is

$$\bar{E}(t) \sim \int_0^L \eta^2(x,t)dx$$

remember that

$$\partial_x\Phi = v_x = c_0\eta$$

and so

$$\eta^2 = \frac{1}{c_0^2}v_x^2 \sim E \qquad + \qquad \underbrace{0(\delta^2)}_{\text{Shallow water approx}}$$

where E is the kinetic energy density . Then, $\bar{E}$ corresponds to the total kinetic energy and must be also constant in time:

$$\begin{aligned}
\dot{E} &\sim 2\int_0^L \eta\dot{\eta}\,dx \overset{kdV}{=} -2\int_0^L \eta\partial^3_{xxx}\eta - 2\int_0^L \eta^2\partial_x\eta \\
&= -2\int_0^L \partial_x[(\eta\partial^2_{xx}\eta) - \frac{1}{2}(\partial_x\eta)^2]dx - \frac{2}{3}\int_0^L \partial_x(\eta^3)dx \\
&= 0 \qquad (8.13)
\end{aligned}$$

- There are even more conserved quantities
- The numerical method used for integration of the KdV eq. should keep these quantities constant in time

8.3.2 Spectral methods

One approximates the solution η by a finite sum of linearly independent functions φ_ℓ:

$$\tilde{\eta}(x,t) = \sum_{\ell=1}^{N} \xi_\ell(t)\varphi_\ell(x) \qquad (8.14)$$

The $\varphi_\ell(x)$ are called *basic functions* or *base*. These could be polynomials , trigonometric functions, etc. If the base is complete, one has

$$\tilde{\eta} \to \eta \qquad \text{if} \quad N \to \infty$$

The equation which we wish to solve numerically (e.g. the KdV eq.) may have the general form

$$\dot{\eta} = F(\eta) \qquad (8.15)$$

Inserting (8.14), one finds

$$R(x,t) = \dot{\tilde{\eta}} - F(\tilde{\eta}), \qquad (8.16)$$

where R is called residual. One can determine the amplitudes ξ_ℓ requiring that the residual is always perpendicular to *all* basic functions. This method ensures that

$$R \rightarrow 0, \qquad \text{if } \tilde{n} \rightarrow \eta$$

and is called *Galerkin method*. Thus

$$\int R\,\varphi_\ell(x)\,dx \stackrel{!}{=} 0 = \int \varphi_\ell \dot{\tilde{\eta}}\,dx - \int \varphi_\ell F(\tilde{\eta})\,dx = \sum_{\ell'}^{N} \int \xi_{\ell'} \varphi_\ell \varphi_{\ell'}\,dx - \int \varphi_\ell F(\tilde{\eta})\,dx$$

or

$$\sum_{\ell'} a_{\ell\ell'} \dot{\xi}_{\ell'} = F_\ell(\xi_1, \ldots, \xi_N) \tag{8.17}$$

with

$$a_{\ell\ell'} = \int \varphi_\ell \varphi_{\ell'}\,dx\,, \qquad F_\ell = \int \varphi_\ell F\,dx$$

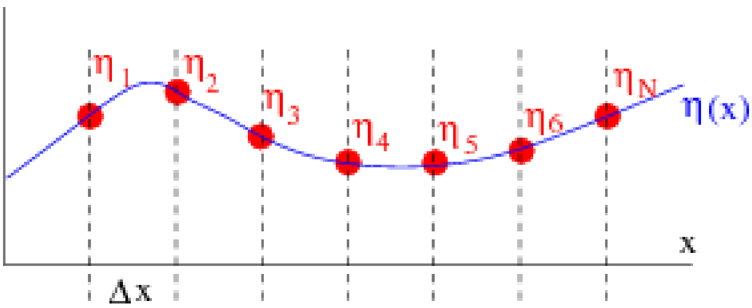
If the φ_ℓ are orthonormal, then $a_{\ell\ell'} = \delta_{\ell\ell'}$ and

$$\dot{\xi}_\ell = F_\ell(\xi_1, \ldots, \xi_N) \qquad \ell = 1 \ldots N \tag{8.18}$$

With (8.18) one has to solve a system of N ordinary differential equations instead of one partial differential equation (8.15).

8.3.3 Finite difference methods

The finite difference methods (FD) compute the solution function only at certain (not always equally spaced) points in real space:



with

$$x_n = n \cdot \Delta x$$

one can write

$$\eta(x,t) = \eta(x_n,t) = \eta_n(t)$$

The spatial derivatives are approximated by finite difference formulas. Here, several orders in Δx can be used, leading to methods with different accuracy, but also with different effort and efficiency. For instance

$$\begin{aligned}
\partial_x \eta(x_n) &\simeq \frac{\eta_{n+1} - \eta_{n-1}}{2\Delta x} \\
\partial^2_{xx} \eta(x_n) &\simeq \frac{\eta_{n+1} - 2\eta_n + \eta_{n-1}}{\Delta x^2} \qquad (8.19) \\
\partial^2_{xxx} \eta(x_n) &\simeq \frac{\eta_{n+2} - 2\eta_{n+1} + 2\eta_{n-1} - \eta_{n-2}}{2\Delta x^3} \qquad (8.20) \\
\text{etc.} &\quad \ldots
\end{aligned}$$

Finally one gets from (8.15) (for the spacial case of the KdV eq.)

$$\dot{\eta}_\ell = F(\eta_{\ell-2}, \eta_{\ell-1}, \eta_\ell, \ldots, \eta_{\ell+2}) \,, \qquad \ell = 1 \ldots N$$

again a system of N ordinary differential equations.

8.3.4 Time integration

The remaining task is to solve a large system of ordinary diff. eqs. of the general form

$$\dot{\eta}_\ell = F_\ell(\eta_1 \ldots \eta_N) \qquad \ell = 0 \ldots N \tag{8.21}$$

This is done by discretizing the time

$$t_n = n \cdot \Delta t$$

with the time step Δt. We use the notation

$$\eta_\ell(t) = \eta_\ell(t_n) \equiv n^n_\ell$$

To approximate the time derivative in (8.21), several possibilities are in order. We concentrate on one-step methods. They have the lowest accuracy but the highest efficiency.

A. Euler forward (explicit)

Taking

$$\dot{\eta}_\ell^n \approx \frac{(\eta_\ell^{n+1} - \eta_\ell^n)}{\Delta t} \tag{8.22}$$

leads to the Euler forward scheme

$$\eta_\ell^{n+1} = \eta_\ell^n + \Delta t \cdot F_\ell(t_n)$$

B. Euler backward (implicit)

with

$$\dot{\eta}_\ell^n \approx \frac{(\eta_\ell^n - \eta_\ell^{n-1})}{\Delta t}$$

one finds the Euler-backward scheme

$$\eta_\ell^n = \eta_\ell^{n-1} + \Delta t \cdot F_\ell(t_n) \tag{8.23}$$

- drawback: (8.23) has to be solved for η_ℓ^n
- advantage: numerical stability can be much better

C. Crank-Nicolson

A combination of forward and backward methods leads to the so-called Crank-Nicolson scheme. It is of higher order in Δt (better accuracy) and reads

$$\eta_\ell^{n+1} = \eta_\ell^n + \frac{1}{2}\Delta t\Big(F_\ell(t_n) + F_\ell(t_{n+1})\Big)$$

D. Leap-frog

Taking instead of (8.22) the formula

$$\dot{\eta}_\ell^n \approx \frac{\eta_\ell^{n+1} - \eta_\ell^{n-1}}{2\Delta t}$$

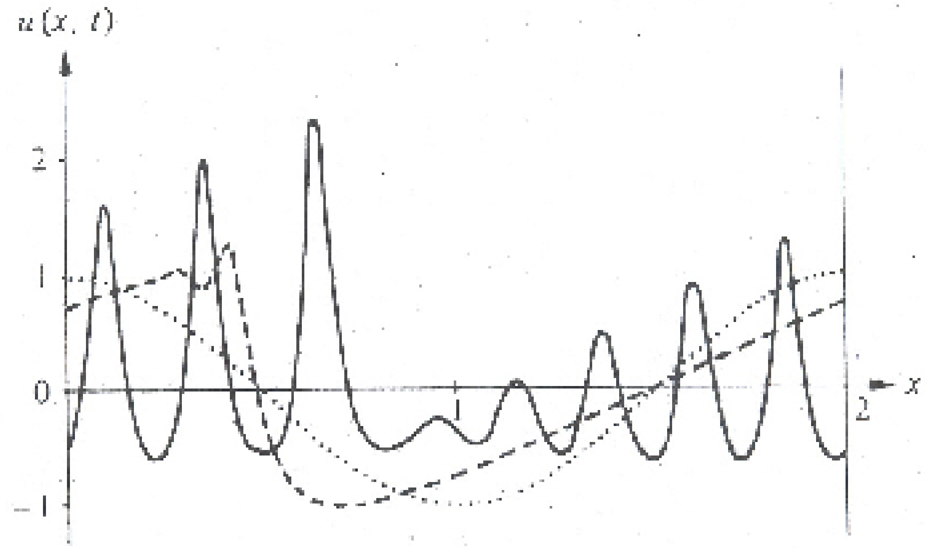
one arrives at the scheme

$$\eta_\ell^{n+1} = \eta_\ell^{n-1} + 2\Delta t F_\ell(t_n)$$

This is called leap-frog method.

8.3.5 Method of Zabusky and Kruskal

The solution of the periodic boundary-value problem for the KdV equation (after Zabusky & Kruskal, 1965). Initial profile at $t = 0$ (dotted line); profile at $t = 1/\pi$ (broken line); profile at $t = 3.6/\pi$ (full line).



Numerically found time evolution as solution of the KdV eq., from P. G. Drazin & R. S. Johnson: Solitons: an introduction

This is a special method to solve the kdV equation:

$$\dot{\eta} = -\partial^3_{xxx}\eta - \eta\partial_x\eta$$

developed in 1965. Use of the Leap-frog method yields:

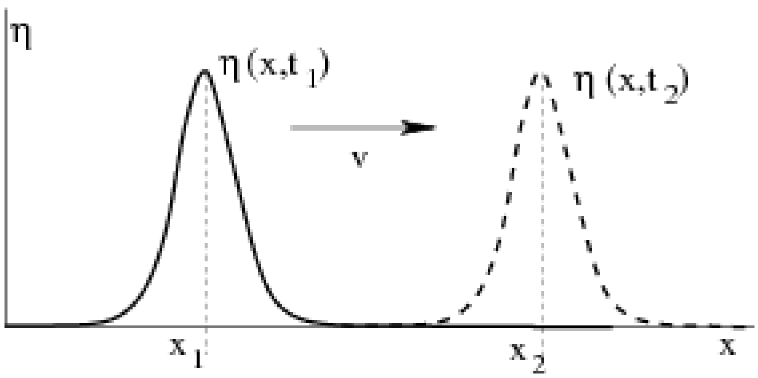
$$
\begin{aligned}
\eta_\ell^{n+1} &= \eta_\ell^{n-1} - \underbrace{\frac{1}{3}\left(\eta_{\ell+1}^n + \eta_\ell^n + \eta_{\ell-1}^n\right)}_{\eta_\ell^n} \cdot \underbrace{\left(\frac{\eta_{\ell+1}^n - \eta_{\ell-1}^n}{\Delta x}\right)}_{2\partial_x \eta^n} \cdot \Delta t \\
&- \underbrace{\left(\frac{\eta_{\ell+2}^n - 2\eta_{\ell+1}^n + 2\eta_{\ell-1}^n - \eta_{\ell-2}^n}{\Delta x^3}\right)}_{2\partial_{xxx}^3 \eta_\ell^n} \cdot \Delta t
\end{aligned}
\tag{8.24}
$$

both, "mass" $\sum_\ell \eta_\ell^n$ and "energy" $\sum_\ell (\eta_\ell^n)^2$ are conserved to $O(\Delta t^2)$!

8.4 Analytical one-soliton solution

In this section we show the classical one-soliton solution which can be found analytically.

We search for a solution which moves with a constant velocity v, say to the right.



$$t_2 > t_1$$

$$\frac{x_2 - x_1}{t_2 - t_1} = v$$

if the shape is constant, then

$$\eta(x,t) = u(x - vt) = u(\tilde{x})$$

$\tilde{x}$ is the coordinate in the co-moving frame with v (Galilei transform)

$$x = \tilde{x} + v\,t, \qquad \tilde{x} = x - v\,t, \qquad t = \tilde{t}$$

For the kdV equation, we express the derivatives in the co-moving frame

$$\partial_x = \partial_{\tilde{x}}, \qquad \partial_t = \underbrace{\left(\frac{\partial \tilde{t}}{\partial t}\right)}_{1} \partial_{\tilde{t}} + \underbrace{\left(\frac{\partial \tilde{x}}{\partial t}\right)}_{-v} \partial_{\tilde{x}} = \partial_{\tilde{t}} - v\,\partial_{\tilde{x}}$$

Using this, the KdV eq. for u reads (we omit all wiggles):

$$(\underbrace{\partial_t}_{=0,} - v\partial_{\tilde{x}})u = -\partial^3_{xxx}u - u\partial_x u$$

Since we are only interested in stationary solutions in the co-moving frame, we put the time derivative to zero. Thus one has (prime denotes derivative by x):

$$vu' - u''' - uu' = 0 \tag{8.25}$$

This is an ordinary DEQ in for $u(x)$. A first integral is easy to find and reads:

$$vu - u'' - \frac{1}{2}u^2 = c \tag{8.26}$$

Where the integration constant c is fixed using boundary conditions $\tilde{x} \to \pm\infty$. One may require (for localized pulses)

$$u, u', u'' \longrightarrow 0 \qquad \text{for} \quad x \longrightarrow \pm\infty$$

this gives $c = 0$ and

$$v\,u - u'' - \frac{1}{2}\,u^2 = 0 \tag{8.27}$$

or

$$u'' = v\,u - \frac{1}{2}\,u^2 \tag{8.28}$$

This is analogue to a one-dimensional motion of a particle in a potential! We identify

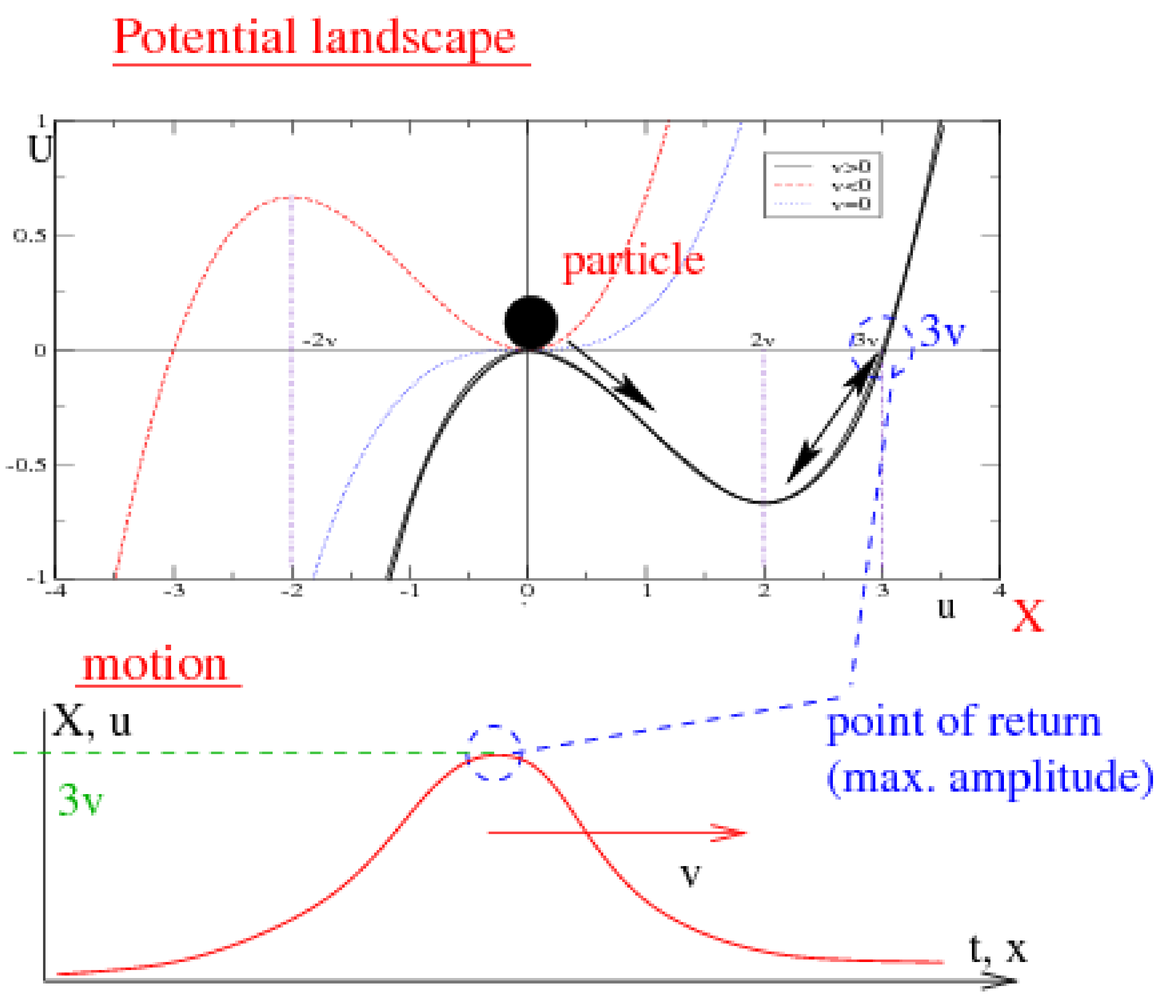
$$x \leftrightarrow t, \qquad u \leftrightarrow X$$

and have

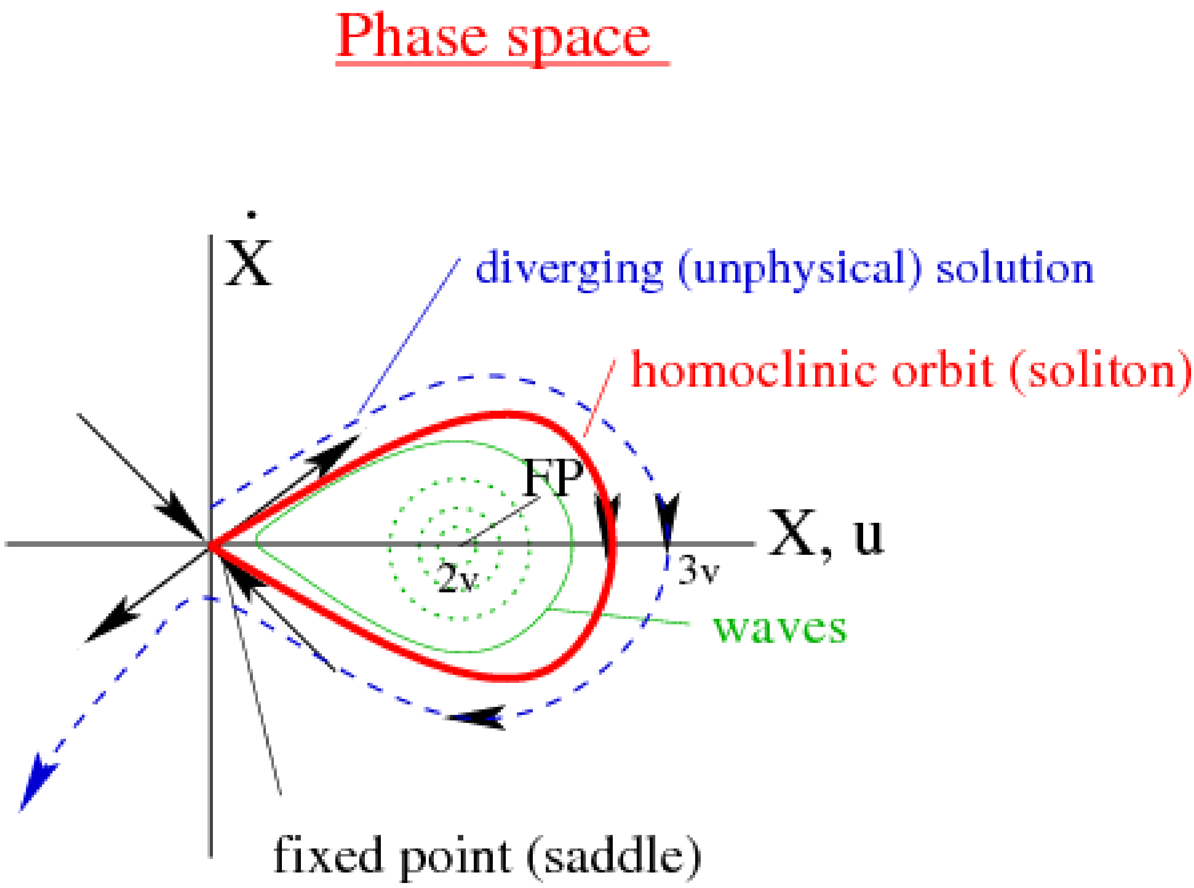
$$\ddot{X} = F(X) = -\frac{dU}{dX}$$

where $U(X)$ serves as a potential function (potential energy). One finds

$$U(X) = -\int F(X)\, dX = -\int (v\, X - \frac{1}{2}\, X^2)\, dX = -\frac{1}{2}\, v\, X^2 + \frac{1}{6}\, X^3$$



amplitude ~ velocity, smaller solitons are slower


Phase space
$\dot{X}$
diverging (unphysical) solution
homoclinic orbit (soliton)
FP
X, u
2v
3v
waves
fixed point (saddle)