Waves and Instabilities

A basic course on hydrodynamics

with geophysical applications

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Preface

These lecture notes are based on a course held at the Brandenburg Technical University Cottbus 2005/06 and 2006/07. The course consists of two hours of lectures weekly, supplemented by two hours of exercises. It has three main parts: (I) Introduction to the continuum description and to the basic hydrodynamic equations, (II) waves in inviscid fluids, and (III) instabilities and pattern formation.

Since the lectures were developed for the master course *Euro Hydro-Informatics and Water Management*, a certain emphasis is layed on geophysical applications as

- water waves in deep and shallow water
- generation of water waves by sea quakes, Tsunamis
- fluid motion in a rotating frame, Coriolis force
- atmospheric motion influenced by Coriolis force and thermal gradients
- Solitons
- Kelvin-Helmholtz instability
- Rayleigh-Bénard thermal convection

It is a pleasure to thank Dr. Rodica Borcia for preparing and performing the exercises. Large parts of the notes were typed in \LaTeX{} by Mrs. Katrin Gregor who also produced most of the figures of the first part.

Cottbus, January 2007

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Part I

Basics
Chapter 1

Kinematics of a Continuum

1.1 Material description and spatial description

- A solid, a fluid or a gas consists of microscopic particles. We wish to describe the motion of $N$ such particles located at $\vec{r}_n$.

\[ \frac{\ddot{\vec{r}}_n(t)}{\dot{\vec{r}}_n(t)} = \vec{F}_N(\vec{r}_1 \ldots \vec{r}_N) \]

$n = 1 \ldots N$

This is in Newtonian mechanics. Other possibilities are to use a quantum mechanical wave function $\Psi(\vec{r}_1 \ldots \vec{r}_N, t)$. This can be important for quantum fluids, but will not be considered in this lectures.

- microscopic picture of the “continuum”

Problems:
- $N \approx 0(10^{23})$
- too complicated
- too much information

Continuum limit solutions:
- mesoscopic theory
- Statistical physics

We consider the continuum limit:
Idea: take infinitely many particles
- particles are no longer countable
- description by \( \vec{r}_n \) makes no sense
- instead of particles, use volume elements

A volume element contains so many particles that microscopic properties are not seen (averaging)

Identification: A volume element is identified by its position \( \vec{R} \) at \( t = t_0 \)
- Reference state, undeformed state –

\[
\vec{R} = \vec{r}(\vec{R}, t = t_0)
\]

Complete description, if \( \vec{r}(\vec{R}, t) \) is known!

\( \vec{r}(\vec{R}, t) \) is a (complicated) transformation with \( t \) as a parameter

It maps \( \vec{R} \rightarrow \vec{r} \)

\( \vec{r}(\vec{R}, t) \) is called “material description” (Lagrangian description)

same for velocity, temperature, etc.
1.2. MATERIAL DERIVATIVE

\[ \vec{v}(\vec{R}, t) \]
\[ T(\vec{R}, t) \]  
Velocity, temperature of the same volume element

More convenient is the “spatial description” (Eulerian description)

\[ T(\vec{r}, t) \]
\[ \vec{V}(\vec{r}, t) \]  
Velocity, temperature at a certain point \( \vec{r} \) and a certain time \( t \), but different volume elements

1.2 Material derivative

How does a certain property of a specified volume element change in course of time?

Answer is easy in material description!

e. g. temperature

\[ \frac{\partial T}{\partial t} = \left( \frac{\partial T}{\partial t} \right)_{\vec{R}} = \frac{D T}{D t} \]

take the same \( \vec{R} \)

follow the volume element

But what is the answer in the spatial description (fixed laboratory frame)?

\[ T(\vec{r}, t) = T(\vec{r}(\vec{R}, t), t) \]

\( \rightarrow \vec{r} \) depends on \( t \).

using the chain rule
\[
\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt} + \frac{\partial T}{\partial z} \frac{dz}{dt}
\]

\[
= \frac{\partial T}{\partial t} + (\bar{v} \cdot \nabla)T
\]  

(1.1)

\[\nabla \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \text{ Nabla vector operator}
\]

The time derivative of any quantity \(A(\bar{r}, t)\) in spatial description has always two parts

\[
\frac{DA}{Dt} = \frac{\partial A}{\partial t} + (\bar{v} \cdot \nabla)A
\]

Change of
Change of
Change of
A of the
A at a
A while
Volume element
fixed position \(\bar{r}\)
moving along
\(\bar{r}(\bar{R}, t)\) with \(\bar{v}(\bar{r}, t)\)

one needs the velocity field

\[\bar{v}(\bar{r}, t)\]

if the continuum is moving.

Another important quantity is the acceleration of a certain volume element

\[
\bar{a}(\bar{R}, t) = \frac{D\bar{v}}{Dt} = \frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla)\bar{v}
\]

\[\text{material spatial}\]
\[\text{description description}\]

dynamics (classical) of a continuum is fixed by Newtons 2nd law:
1.3. DISPLACEMENT FIELD

\[ m \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = \vec{f}(\vec{r}, t) \]

\( \vec{f} \): (inner and outer) forces at position \( \vec{r} \), time \( t \)

- Fluid mechanics is (almost) always a nonlinear theory

  but for small velocities:

  \[
  \frac{D\vec{v}}{Dt} \approx \frac{\partial \vec{v}}{\partial t}
  \]

  - This approximation is only good close to walls or obstacles (boundary layers), or in microfluidics. For large velocities, Turbulence occurs which is clearly a non-linear phenomenon. Also instabilities and pattern formation exist due to nonlinearities.

1.3 Displacement field

up to here: description by trajectories of volume elements (like particles).

\[ \vec{r} = \vec{r}(\vec{R}, t) \]

\[ \vec{S}(\vec{R}, 0) \]

Figure 1.1:

now: consider the displacement \( \vec{S}(\vec{R}, t) \)

- property of a certain volume element
\[ \vec{r}(\vec{R}, t) = \vec{R} + \vec{S}(\vec{R}, t) \]

for small displacement we have \( \vec{S}(\vec{R}, t) = \vec{S}(\vec{r}, t) \)

proof:

\[ \vec{S}(\vec{r}, t) = \vec{S}(\vec{R} + \vec{S}, t) \quad \text{Taylor} \quad \vec{S}(\vec{R}, t) + \nabla \vec{S} \bigg|_{\vec{R}} \cdot \vec{S} \]

(1.2)

Example: uniaxial compression

\[ x = X, \quad y = Y, \quad z = \frac{Z}{2} \]

Compute the displacement field.

Answer:

\[ \vec{S} = \vec{r} - \vec{R} = \begin{pmatrix} 0 \\ 0 \\ -\frac{Z}{2} \end{pmatrix} \]

Example: shearing motion

\[ x = X + k \cdot t \cdot Y, \quad y = Y, \quad z = Z \]

\[ \vec{S} = \begin{pmatrix} k \cdot t \cdot Y \\ 0 \\ 0 \end{pmatrix} \]
1.4 Infinitesimal deformations

consider two neighboring points $P$ and $Q$.

\[ P : \quad \vec{r} = \vec{R} + \vec{S}(\vec{R}) \]
\[ Q : \quad \vec{r} + d\vec{r} = \vec{R} + d\vec{R} + \vec{S}(\vec{R} + d\vec{R}) \]
\[ Q - P : \quad d\vec{r} = d\vec{R} + \underbrace{\vec{S}(\vec{R} + d\vec{R})}_{\text{Taylor}} \]

\[ d\vec{r} = d\vec{R} + d\vec{R}(\nabla \circ \vec{S}) = d\vec{S} \]

$d\vec{r}$ and $d\vec{S}$ have different directions $\rightarrow (\nabla \circ \vec{S})$ is a tensor!
CHAPTER 1. KINEMATICS OF A CONTINUUM

\[ \nabla \circ \vec{S} = \begin{pmatrix} \frac{\partial S_x}{\partial x} & \frac{\partial S_y}{\partial x} & \frac{\partial S_z}{\partial x} \\ \frac{\partial S_x}{\partial y} & \frac{\partial S_y}{\partial y} & \frac{\partial S_z}{\partial y} \\ \frac{\partial S_x}{\partial z} & \frac{\partial S_y}{\partial z} & \frac{\partial S_z}{\partial z} \end{pmatrix} \]  \hspace{1cm} (1.3)

dyadic product of \( \nabla \) and \( \vec{S} \) in components:

\[ \frac{\partial S_j}{\partial x_i} \]

displacement gradient, distorsion tensor

\[ \bar{\beta} = \nabla \circ \vec{S}, \quad \beta_{ij} = \frac{\partial S_j}{\partial x_i} \]

\( \bar{\beta} \) assigns to two points with distance \( d\vec{r} \) the relative displacement \( d\vec{S} \). It describes completely an infinitesimal deformation!

\[ d\vec{S} = d\vec{r} \cdot \bar{\beta} \]

In components

\[ dS_i = \sum_{j=1}^{3} \beta_{ji} dr_j \]  \hspace{1cm} (1.4)

In this script we use Einstein’s sum convention. This means we drop the sum and write instead of (1.4) the short form

\[ dS_i = \beta_{ji} dr_j \]

where over indices which occur twice on one side of an equation the sum runs from one to three.

1.5 A short paragraph on tensors

1.5.1 Definition as a linear transformation

Let \( T \) be a transformation (operation) which transforms any vector into another vector:

\[ T \cdot \vec{a} = \vec{b}, \]  \hspace{1cm} (1.5)
1.5. A SHORT PARAGRAPH ON TENSORS

if $T$ has the linear properties

$$ T \cdot (\vec{a}_1 + \vec{a}_2) = T \cdot \vec{a}_1 + T \cdot \vec{a}_2, \quad \vec{a}_1, \vec{a}_2 \quad \text{arbitrary} $$

$$ T \cdot (\alpha \cdot \vec{a}) = \alpha \cdot T \cdot \vec{a} \quad \alpha, \vec{a} \quad \text{arbitrary} $$

then $T$ is a second order tensor (linear transformation).

If for all $\vec{a}$

$$ T \cdot \vec{a} = S \cdot \vec{a} $$

then

$$ T = S $$

1.5.2 Components of a tensor

The components of a Tensor depend on the base vectors $\hat{e}_i$.

For vectors:

$$ \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \text{is short for} \quad \vec{a} = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3 $$

Multiplying from the left with $\hat{e}_i$ yields (if $(\hat{e}_i \cdot \hat{e}_j) = \delta_{ij}$)

$$ \vec{a} \cdot \hat{e}_i = a_i $$

Now we use the special orthogonal base

$$ \hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} $$
Writing down (1.5) in components:

\[ T_{11}a_1 + T_{12}a_2 + T_{13}a_3 = b_1 \]
\[ T_{21}a_1 + T_{22}a_2 + T_{23}a_3 = b_2 \]
\[ T_{31}a_1 + T_{32}a_2 + T_{33}a_3 = b_3 \]

and using \( \vec{a} = \hat{e}_1 \) gives

\[
\mathbf{T} \cdot \hat{e}_1 = \begin{pmatrix} T_{11} \\ T_{21} \\ T_{31} \end{pmatrix} = T_{11}\hat{e}_1 + T_{21}\hat{e}_2 + T_{31}\hat{e}_3
\] (1.6)

or in general

\[
\mathbf{T} \cdot \hat{e}_j = T_{1j}\hat{e}_1 + T_{2j}\hat{e}_2 + T_{3j}\hat{e}_3 = T_{kj}\hat{e}_k
\] (1.7)

and multiplying with \( \hat{e}_i \) finally

\[
\hat{e}_i \cdot \mathbf{T} \cdot \hat{e}_j = T_{kj}(\hat{e}_i \hat{e}_k) = T_{ij}
\] (1.8)

In the same way as

\[ a_i = \vec{a} \cdot \hat{e}_i \]

we may write

\[ T_{ij} = \hat{e}_i \cdot \mathbf{T} \cdot \hat{e}_j \]

\( T_{ij} \) is called “the matrix of the tensor \( \mathbf{T} \)”. 

but remember: with respect to the base vectors \( \hat{e}_i \)
1.5.3 Symmetric tensors

If

\[ T \cdot \vec{a} = \vec{a} \cdot T \quad \text{then} \quad T_{ij} = T_{ji} \]

and \( T \) is a symmetric tensor.

1.5.4 Sum of two tensors

if

\[ T \cdot \vec{a} + S \cdot \vec{a} = W \cdot \vec{a} \]

for all \( \vec{a} \), then

\[ W = T + S \]

is the sum of \( T \) and \( S \)

in components

\[ W_{ij} = T_{ij} + S_{ij} \]

1.5.5 Product of two tensors

(a) Inner product

Let \( (T \cdot S) \cdot \vec{a} = T \cdot (S \cdot \vec{a}) \)

and \( (S \cdot T) \cdot \vec{a} = S \cdot (T \cdot \vec{a}) \),

for arbitrary \( \vec{a} \), then \( (T \cdot S) \) and \( (S \cdot T) \) are called (inner) product of \( T \) and \( S \) (Obviously there are two possibilities).
The components of the inner product read:

\[(\mathbf{T} \cdot \mathbf{S})_{ij} = \hat{e}_i \cdot (\mathbf{T} \cdot \mathbf{S}) \cdot \hat{e}_j = \hat{e}_i \cdot \mathbf{T} (\mathbf{S} \cdot \hat{e}_j) = T_{ik} S_{kj}\]

\[= T_{ik} S_{kj}\]  \hspace{1cm} (1.9)

and similarly

\[(\mathbf{S} \cdot \mathbf{T})_{ij} = S_{ik} T_{kj}\]  \hspace{1cm} (1.10)

in general:

\[\mathbf{S} \cdot \mathbf{T} \neq \mathbf{T} \cdot \mathbf{S}\]

the tensor product is not commutative!

(compare to inner (scalar) product of two vectors (scalar valued = independent on \(\hat{e}_i\)).

\[\vec{a} \cdot \vec{b} = (a_i \hat{e}_i) (b_j \hat{e}_j) = a_i b_j (\hat{e}_i \cdot \hat{e}_j) = a_i b_i \]

(b) outer product

Def:

\[T_{ij} S_{kl} = Q_{ijkl}\]

is the outer product of \(\mathbf{T}\) and \(\mathbf{S}\). Then, \(\mathbf{Q}\) is a fourth order tensor

Compare to outer product of two vectors!

\[a_i \circ b_j = T_{ij}, \quad \vec{a} \circ \vec{b} = \mathbf{T}\]

\(\mathbf{T}\) is a tensor (second order).

\[\rightarrow \text{“dyadic product”}\]
1.5. A SHORT PARAGRAPH ON TENSORS

1.5.6 Contraction and trace of a tensor

The contraction of a tensor is the sum over two indices. This reduces its order by two.

Example:

$$\sum_j Q_{ijjm} = Q_{ijjm} = P_{im}$$

$P$ is the (one of the) contraction (s) of $Q$

The trace of a tensor is the contraction of a second order tensor, what leaves a scalar.

$$T_{ii} = \text{tr}(T), \quad \text{trace of } T$$
$$\text{tr}(T + S) = \text{tr}(T) + \text{tr}(S)$$
$$\text{tr}(\vec{a} \circ \vec{b}) = \vec{a} \cdot \vec{b}$$

1.5.7 Antisymmetric Tensors

if $T_{ij} = T_{ji}$ then $T$ is a symmetric tensor

if $T_{ij} = -T_{ji}$ then $T$ is an antisymmetric tensor

antisymmetric tensors of 2nd order can have only 3 independent elements

$$T^A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

and clearly

$$T^A_{ii} = 0, \quad \text{and } \text{tr}(T^A) = 0$$

Important third order antisymmetric tensor: $\varepsilon$ tensor
CHAPTER 1. KINEMATICS OF A CONTINUUM

\[ \varepsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k = 1, 2, 3 \text{ or even permutations} \\ -1 & \text{if } i, j, k \text{ are odd permutations of } 1, 2, 3 \\ 0 & \text{if } 2 \text{ or } 3 \text{ indices are equal} \end{cases} \]

\( \varepsilon \) can be used to compute the vector product (don’t confuse with the outer product)

\[ \vec{a} = \vec{b} \times \vec{c} \implies a_i = \varepsilon_{ijk} b_j c_k \]

Each tensor can be decomposed:

\[ T = T^S + T^A \]

with the symmetric tensor

\[ T^S = \frac{1}{2}(T + T^T) \]

and the antisymmetric tensor

\[ T^A = \frac{1}{2}(T - T^T) \]

\( T^T \) denotes the transpose of \( T \):

\[ T^T_{ij} = T_{ji} \]

1.5.8 The dual vector

Let \( T^A \) be an antisymmetric 2nd order tensor. Then its dual vector is defined as

\[ \vec{T}^A = \begin{pmatrix} -T^A_{23} \\ T^A_{31} \\ T^A_{12} \end{pmatrix} = -\frac{1}{2} \varepsilon_{ijk} T^A_{jk} \hat{e}_i \] (1.11)

and
1.5. A SHORT PARAGRAPH ON TENSORS

\[ T^A \cdot \vec{a} = \vec{r}^A \times \vec{a}, \quad \perp \text{ on } \vec{a} \text{ and } \vec{r}^A \]

can be used to compute the vector product.

1.5.9 Eigenvalues and eigenvectors of a tensor

Definition:

If

\[ T \cdot \vec{a} = \lambda \vec{a}, \quad \lambda \in C \]

then \( \vec{a} \) is an eigenvector of \( T \)

\( \lambda \) is the eigenvalue of \( T \) that belongs to \( \vec{a} \)

To compute the eigenvalues and eigenvectors one has to solve a linear homogeneous system

\[ (T - 1 \cdot \lambda) \cdot \vec{a} = 0 \]

This has nontrivial solutions (\( \vec{a} \neq 0 \)), only if the solvability condition

\[ \det (T - 1\lambda) = 0 \]

is fulfilled. This leads to the characteristic equation (polynomial in \( \lambda \))

\[ \lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0 \]

It has three (in general complex) roots:

\[ \lambda_1, \lambda_2, \lambda_3 \]

From that one may compute the eigenvectors

\( \vec{a}_1, \vec{a}_2, \vec{a}_3 \)
CHAPTER 1. KINEMATICS OF A CONTINUUM

A symmetric tensor has three real valued eigenvalues. Its eigenvectors \( \vec{a}_1, \vec{a}_2, \vec{a}_3 \) are mutually orthogonal. They can be normalized, \( |\vec{a}_i| = 1 \) and form a system of orthogonal base vectors

\[
\vec{a}_i \cdot \vec{a}_j = \delta_{ij}
\]

\( \lambda_i \) are independent from the base \( \hat{e}_i \). From that it follows that \( I_1, I_2, I_3 \) are also independent.

They are called scalar invariants of \( T \)

\[
I_1 = T_{11} + T_{22} + T_{33} = \text{tr} (T) \tag{1.12}
\]

\[
I_2 = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix} = \frac{1}{2} \left[ (\text{tr} (T))^2 - \text{tr} (T^2) \right] \tag{1.13}
\]

\[
I_3 = \det (T) \tag{1.14}
\]

or, in terms of \( \lambda_i \)

\[
I_1 = \lambda_1 + \lambda_2 + \lambda_3 \\
I_2 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 \\
I_3 = \lambda_1 \lambda_2 \lambda_3
\]

1.5.10 Differential operators, scalar-, vector-, tensor-fields

(a) Gradient of a scalar field \( \Phi(\vec{r}) \)

How changes \( \Phi \) if \( \vec{r} \) is changed?

\[
\frac{d\Phi}{d\vec{r}} = \Phi(\vec{r} + d\vec{r}) - \Phi(\vec{r}) \equiv \nabla \Phi \cdot \frac{d\vec{r}}{\text{vector}} \tag{1.15}
\]

Taylor expansion:
1.5. A SHORT PARAGRAPH ON TENSORS

\[ \Phi(\vec{r} + d\vec{r}) - \Phi(\vec{r}) = \partial_x \Phi dx + \partial_y \Phi dy + \partial_z \Phi dz + O(dr^2) \]

\[ = \nabla \Phi \cdot d\vec{r} \]

\[ \rightarrow \nabla \Phi = \text{grad} \Phi = \begin{pmatrix} \partial_x \Phi \\ \partial_y \Phi \\ \partial_z \Phi \end{pmatrix} \quad (1.16) \]

directional derivative:

\[ \frac{\partial \Phi}{\partial \vec{n}} = \vec{n} \cdot \nabla \Phi \quad (1.17) \]

This is a scalar quantity which measures the change of \( \Phi \) if \( \vec{r} \) goes to \( \vec{r} + d\vec{r} \) and \( d\vec{r} \parallel \vec{n} \)

(b) Gradient of a vector field \( \vec{V}(\vec{r}) \) (outer product)

\[ \frac{d\vec{V}}{\text{vector}} = \vec{V}(\vec{r} + d\vec{r}) - \vec{V}(\vec{r}) \equiv \frac{d\vec{r}}{\text{vector}} \cdot (\nabla \circ \vec{V}) \quad (1.18) \]

in general: \( d\vec{V} \) and \( d\vec{r} \) are not parallel!

Taylor expansion (in components)

\[ V_i(\vec{r} + d\vec{r}) - V_i(\vec{r}) = \partial_x V_i dx + \partial_y V_i dy + \partial_z V_i dz \]

\[ = \frac{\partial V_i}{\partial x_j} dx_j \]

\[ (\nabla \circ \vec{V})_{ij} = \frac{\partial V_j}{\partial x_i} \quad \text{dyadic product of } \nabla \text{ and } \vec{V} \quad (1.19) \]

(c) Divergence of a vector field (inner product)

\[ \nabla \cdot \vec{V} \equiv \text{div} \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = Q(\vec{r}) \quad (1.20) \]

\( Q(\vec{r}) \) denotes sources and sinks of \( \vec{V}(\vec{r}) \)!
1.5.11 Defining tensors by transformation laws

The laws of nature are independent of (the orientation of) the coordinate systems.

Transformation

\[ x_i = x_i(x'_1, x'_2, x'_3) \]
\[ x'_i = x'_i(x_1, x_2, x_3) \]

The Jacobi matrix is defined as

\[ A_{ij} = \frac{\partial x_i}{\partial x'_j} \]

Physical objects are transformed by certain rules, depending on their vector character:
1.6. **DECOMPOSITION OF THE DISTORTION TENSOR**

1. Scalars (mass, charge, density, temperature, ...)

\[ S(x_i) = S'(x'_i) \]

2. Vectors (forces, velocity, acceleration, heat flow, ...)

\[ V_i = A_{ij} V'_j \]

3. Tensors (stress, deformation, ...)

\[ T_{ij} = A_{ik} A_{jl} T'_{kl} \]

### 1.6 Decomposition of the distortion tensor

We first write the distortion tensor as a sum of a symmetric and an antisymmetric tensor and then interpret the physical meaning of the two parts.

\[
\nabla \circ \vec{S} = \beta = \varepsilon + \varphi \quad (1.21)
\]

\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial S_{ij}}{\partial x_j} + \frac{\partial S_{ij}}{\partial x_i} \right) \text{ symmetric strain tensor} \quad (1.22)
\]

\[
\varphi_{ij} = \frac{1}{2} \left( \frac{\partial S_{ij}}{\partial x_j} - \frac{\partial S_{ij}}{\partial x_i} \right) \text{ antisymmetric rotation tensor} \quad (1.23)
\]

– Geometrical interpretation of the decomposition
CHAPTER 1. KINEMATICS OF A CONTINUUM

\[ \text{distortion} = \text{strain} + \text{rotation} \]

(a) Symmetric part

\( \varepsilon \) describes volume dilatation (compression) and shear

Change of volume:

\[ \frac{\Delta V}{V} = \text{tr} \beta = \text{tr} \varepsilon = \text{div} \vec{s} \] (1.24)

(b) Antisymmetric part

\( S_{1y} = \alpha_1 X \)

\( S_{2x} = -\alpha_2 Y \)

pure rotation: \( \alpha_1 = \alpha_2 = \alpha \)

\[ \begin{align*}
\beta_{xy} &= \frac{\partial S_y}{\partial x} = \alpha_1 = \alpha \\
\beta_{yx} &= \frac{\partial S_x}{\partial y} = -\alpha_2 = -\alpha \\
\phi_{xy} &= \frac{1}{2} \left( \frac{\partial S_y}{\partial x} - \frac{\partial S_x}{\partial y} \right) = \alpha \\
\phi_{yx} &= -\phi_{xy} = -\alpha
\end{align*} \] (1.25)

The components of the antisymmetric part are the angles of (infinite) rotations:

1. \( \phi_{xy} \): rotation with respect to z-axis
2. \( \phi_{yz} \): rotation with respect to x-axis
3. \( \phi_{zx} \): rotation with respect to y-axis
Chapter 2

Forces, Deformation and Stress

Solids

We start considering an elastic material following Hooke’s law. In equilibrium there must be a balance between external forces and internal stress:

As an example take an elastic bar, fixed on one side. If gravitation forces act on the bar it will deform until equilibrium is reached.
Fluids

Now turn to viscous fluids. Again, external forces (pressure) must be balanced by internal stress. These stresses originate from volume dilatation (elastic) and viscosity (shear):

Example: Flow through a pipe. The pressure gradient accelerates the fluid until viscous forces are in balance.
2.1 Stress vector and stress tensor

2.1.1 Stress vector

The stress vector is defined as a force per area element:

\[ \vec{t} = \lim_{\Delta A} \frac{\Delta \vec{F}}{\Delta A} = \left[ \frac{N}{m^2} \right] \]

(Force / Area)

Cauchy’s Stress principle:

\[ \vec{t} = \vec{t}(\vec{r}, \hat{n}, t) \]

It says that the stress depends on the direction of \( \hat{n} \), but not on curvature, etc.

2.1.2 Stress tensor

In general, \( \vec{t} \) and \( \hat{n} \) are not parallel!
This can be seen from the following picture:

tetrahedron with the four sides 
\( \Delta A, \Delta A_1, \Delta A_2 \)

\[
\Delta A_i = n_i \Delta A
\]

(2.1)

In equilibrium, the forces on each side have to balance:

\[
\vec{t} \Delta A + \vec{t}_1 \Delta A_1 + \vec{t}_2 \Delta A_2 + \vec{t}_3 \Delta A_3 = 0
\]

(2.2)

this can be solved for the stress vector

\[
\vec{t} = -\vec{t}_1 n_1 - \vec{t}_2 n_2 - \vec{t}_3 n_3
\]

(2.3)

what reads in components:

\[
\begin{pmatrix}
  t_x \\
t_y \\
t_z
\end{pmatrix}
= -
\begin{pmatrix}
  t_{1x}, & t_{2x}, & t_{3x} \\
t_{1y}, & t_{2y}, & t_{3y} \\
t_{1z}, & t_{2z}, & t_{3z}
\end{pmatrix}
\cdot
\begin{pmatrix}
  n_x \\
n_y \\
n_z
\end{pmatrix}

= -\overline{T} \cdot \hat{n}
\]

(2.4)

Def: \( \overline{T} \) is the stress tensor.

- if you want to know the stress (force/area) at a certain point, multiply the normal vector through that point with the stress tensor \( \overline{T} \). This results in a vector, namely the stress vector.

Components of \( \overline{T} \):
2.1. STRESS VECTOR AND STRESS TENSOR

\[ T_{ij} = \hat{e}_i \cdot \mathbf{T} \cdot \hat{e}_j \]  

(2.5)

is the i-component of the stress vector \( \mathbf{T} \) at the surface with the normal \( \hat{e}_j \).

2.1.3 Diagonal and off-diagonal components of the stress tensor

Consider a cuboid:

The diagonal components of \( T \) account for stresses perpendicular to the surfaces of the cuboid, they are called “normal stresses”. In a solid (and in a fluid) they provoke volume compression.

\[ T_{ii} = \text{normal stresses} \]

\[ T_{ij} = \text{shear stresses}, \ i \neq j, \]

The off-diagonal components describe stresses tangential to the surfaces. They are called “shear stresses” and cause a shear motion or a shearing of the cuboid.
in fluids: normal stress $\iff$ pressure

(Newton) shear stress $\iff$ friction

### 2.1.4 Symmetry of the stress tensor

In solids (and in fluids), there is usually no torque on a single volume element. Otherwise it would start to rotate. Then,

$$T_{31} \Rightarrow T_{13} \overset{\sim}{=} T_{13} = T_{31}$$

in general:

$$T_{ij} = T_{ji}$$

and $\underline{T}$ is a symmetric tensor.

### 2.2 Stress and forces

if $\underline{T}$ is constant, there is no force on a volume element.

$$\hat{t}_a = -\hat{n}_a \cdot \hat{n}_a$$

$$\hat{t}_b = -\hat{n}_b \cdot \hat{n}_b = \hat{n}_a \cdot \hat{n}_a$$

$$\Delta \vec{t} = \vec{t}_a + \vec{t}_b = 0$$
2.2. STRESS AND FORCES

In general, $T$ depends on $\vec{r}$ (and time)

$$ T = T(\vec{r}, t) $$

now:

$$ \vec{t}_a = -T(\vec{r}_a) \cdot \hat{n}_a $$
$$ \vec{t}_b = -T(\vec{r}_b) \cdot \hat{n}_b = T(\vec{r}_b) \cdot \hat{n}_a \quad (2.6) $$

and:

$$ \Delta \vec{t} = \vec{t}_a + \vec{t}_b = \left[ T(\vec{r}_b) - T(\vec{r}_a) \right] \cdot \hat{n}_a \quad (2.7) $$

~ gradient of stress

Next, we wish to compute the total force acting on a small cuboid, if the stress tensor depends on space. To this end we consider

$$ \Delta V = \Delta x \Delta y \Delta z $$

The $x$-component of the total force is the sum of the $x$-components of the forces acting on the six surfaces:

$$ \Delta F_x = F_x(x + \Delta x, y, z) + F_x(x, y, z) \quad yz-plane $$
$$ + \quad F_x(x, y + \Delta y, z) + F_x(x, y, z) \quad xz-plane $$
$$ + \quad F_x(x, y, z + \Delta z) + F_x(x, y, z) \quad xy-plane $$

$$ = (T_{xx}(x + \Delta x, y, z) - T_{xx}(x, y, z)) \cdot \Delta y \cdot \Delta z $$
$$ + \quad (T_{yx}(x, y + \Delta y, z) - T_{yx}(x, y, z)) \cdot \Delta x \cdot \Delta z $$
$$ + \quad (T_{zx}(x, y, z + \Delta z) - T_{zx}(x, y, z)) \cdot \Delta x \cdot \Delta y $$
\[ \Delta F_x = \left\{ \frac{T_{xx}(x + \Delta x, y, z) - T_{xx}(x, y, z)}{\Delta x} + \frac{T_{yx}(x, y + \Delta y, z) - T_{yx}(x, y, z)}{\Delta y} + \frac{T_{zx}(x, y, z + \Delta z) - T_{zx}(x, y, z)}{\Delta z} \right\} \Delta V \]

\[ \Delta F_x = \left( \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z} \right) \Delta V = (\text{div} \, T)_x \Delta V \]

One introduces the force volume density

\[ \vec{f} \equiv \frac{\Delta \vec{F}}{\Delta V} \]  

(2.8)

Finally one gets

\[ \vec{f}^{(\text{in})} = \text{div} \, T \]

– This shows us how internal forces can be computed by the stress tensor
– The sources and sinks of \( T \) correspond to forces acting on \( \Delta V \) at position \( \vec{r} \)
– In equilibrium, external forces + Internal forces = 0

\[ \vec{f}^{(\text{in})} = -\vec{f}^{(\text{ext})} \]

\[ \text{div} \, T + \vec{f}^{(\text{ext})} = 0 \]  

(2.9)

external forces create sources or sinks of \( T \) (and vice versa)

Remember:
2.2. STRESS AND FORCES

The basic law (2.9) describes the dotted connection.
Chapter 3

The Euler Equations

In this chapter we derive the basic equations that describe the spatio-temporal evolution of a so-called perfect or inviscid fluid, a fluid with no friction or zero viscosity. To this end we shall use eq. (2.9) and express the stress tensor by the pressure. Also inertial terms have to be added to account for acceleration of a volume element.

3.1 Preliminaries

If a fluid is in motion its displacement field and relative displacement field $\Delta \vec{S}$, $\vec{S}$ can get arbitrarily large

- concept of displacement field makes no sense
- fluids are better described by the velocity field

$\vec{v}(\vec{r}, t)$
• We can still use the balance equation (2.9) for forces. We add acceleration:

$$\text{div} \mathbf{T} + \mathbf{f} = \rho \ddot{\mathbf{a}} \quad (3.1)$$

$\ddot{\mathbf{a}}$: acceleration of a certain volume element (Lagrange picture).

To close the description one has to find

$$\mathbf{T} = \mathbf{T}(p) \quad \text{and later} \quad \mathbf{T} = \mathbf{T}(p, \mathbf{v})$$

where $p$ denotes the pressure.

### 3.2 Conservation laws and basic equations

#### 3.2.1 Conserved quantities

There are five important conserved quantities in nature. These are

- momentum $\mathbf{p} \Rightarrow p_x, p_y, p_z$
- total mass $M$
- total energy $E$

A major task of theoretical physics is to find the conservation laws and to derive from them equations of motion. In the case of a fluid, these equations are called hydrodynamic basic equations and describe the spatio-temporal evolution of the five variables

<table>
<thead>
<tr>
<th>velocity</th>
<th>$v_x, v_y, v_z$</th>
<th>$\mathbf{p}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>density</td>
<td>$\rho$</td>
<td>$M$</td>
</tr>
<tr>
<td>temperature</td>
<td>$T$</td>
<td>$E$</td>
</tr>
</tbody>
</table>

Next we describe how one can derive the basic equations for

$$\mathbf{v}(\mathbf{r}, t), \quad \rho(\mathbf{r}, t), \quad T(\mathbf{r}, t)$$

from the conserved quantities $\mathbf{p}$, $\rho$, and $E$. 

3.2.2 Global conservation versus local conservation

Let $Q$ be a conserved quantity in the arbitrary volume $V$, closed by the surface $F(V)$.

\[ \dot{Q}, V \times F(v) \; d^2 f \]

\[ \vec{j} : \text{flux of } Q \text{ through the surface of } V : \]

\[
\frac{d}{dt} Q = - \oint_{F(v)} \vec{j} d^2 \vec{f} \\
\text{global conservation law}
\]

Let $\rho(\vec{r}, t)$ be the density of $Q$:

\[ Q = \int_v \rho \; d^3 \vec{r} \]

\[
\frac{d}{dt} \int_v \rho \; d^3 \vec{r} + \int_{F(v)} \vec{j} d^2 \vec{f} = 0 \\
\downarrow \quad V = \text{const} \quad \downarrow \quad \text{Theorem of Gauß} \\
\int_v \left[ \rho + \text{div} \vec{j} \right] d^3 \vec{r} = 0
\]

now $V$ is arbitrary:

\[
\rho + \text{div} \vec{j} = 0 \\
\text{local conservation law}
\]
3.2.3 Velocity

Instead of using the global conservation of the total momentum (this would be also a possible way, see Landau, Lifschitz), we take the fundamental force balance (3.1) and insert acceleration (inertia) according to:

$$\vec{a} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v}$$  \hspace{1cm} (3.2)

leading to

$$\rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = \text{div} \, \vec{T} + \vec{f}$$

inertial force = inner + outer forces  \hspace{1cm} (3.3)

This describes already the evolution of $\vec{v}$. But how can we compute the inner forces $\text{div} \, \vec{T}$?

In a perfect fluid, there is no shear force. Only normal stresses can occur:

$T_{xx} = T_{yy} = T_{zz} = -p(\vec{r},t)$  \hspace{1cm} (3.4)

or

$$T_{ij} = -p(\vec{r},t) \, \delta_{ij}$$  \hspace{1cm} (3.5)
3.2. CONSERVATION LAWS AND BASIC EQUATIONS

From that one finds the inner forces:

\[
\begin{align*}
\text{(div} T)\text{j} & = \sum_i \partial_i T_{ij} \\
& = -\sum_i \delta_{ij} \partial_i p = -\partial_j p
\end{align*}
\]

\[
\text{inviscid fluid}
\]

\[
\text{div} T = -\text{grad} p
\]

and finally:

\[
\rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = -\text{grad} p + \vec{f}
\]

Euler equations for perfect fluids

(Leonhard Euler, 1707-1783) The Euler eqs. read in components:

\[
\begin{align*}
\rho (\dot{v}_x + v_x \partial_x v_x + v_y \partial_y v_x + v_z \partial_z v_x) & = -\partial_x \rho + f_x \\
\rho (\dot{v}_y + v_x \partial_x v_y + v_y \partial_y v_y + v_z \partial_z v_y) & = -\partial_y \rho + f_y \\
\rho (\dot{v}_z + v_x \partial_x v_z + v_y \partial_y v_z + v_z \partial_z v_z) & = -\partial_z \rho + f_z
\end{align*}
\]

and provide three equations for \( v_x, v_y, v_z \). To find a solution, one must specify \( p = p(\rho, T) \) in form of a material law (state equation)

e. g. for a perfect gas:

\[
p(\rho, T) = \frac{R}{m_0} \rho T, \quad m_0 = \text{molar mass}
\]
3.2.4 Density

Consider a certain mass flux

\[ J = \int_A \rho \vec{v} d^2 \vec{r} \]

through a given surface \( A \), measured in kg/s.

The change of the total mass inside a volume \( V \)

\[ \frac{dM}{dt} = \frac{d}{dt} \int_V \rho d^3\vec{r} = \int_V \dot{\rho} d^3\vec{r} \]  \hspace{1cm} (3.8)

must be equal to the flux through its surface \( F(V) \)

\[ = - \int_{F(V)} \vec{j} d^2 \vec{f} = - \int_{F(V)} \rho \vec{v} d^2 \vec{f} = - \int_V \text{div} (\rho \vec{v}) d^3\vec{r} \]  \hspace{1cm} (3.9)

Gauss theorem

\[ \dot{\rho} + \text{div} (\vec{v} \rho) = 0 \]  \hspace{1cm} (3.10)

Continuity equation

This provides another basic equation for \( \rho(\vec{r},t) \).
3.2. **CONSERVATION LAWS AND BASIC EQUATIONS**

Now we consider the special but very common case of incompressible fluids (at least for not too large pressure gradients, most fluids can be assumed to be incompressible in a good approximation. However, this is usually not the case for gases).

But even in an incompressible fluid, the density can still be a function of time and space (e.g., in complex fluids like mixtures).

but: density of a certain volume element must be constant:

\[
\frac{d\rho}{dt} = 0
\]

Euler picture:

\[
\frac{d\rho}{dt} = \dot{\rho} + \vec{v} \text{grad} \rho = 0, \quad \Rightarrow \quad \dot{\rho} = -\vec{v} \text{grad} \rho \tag{3.11}
\]

→ density inhomogeneities are transported with the flow.

The continuity equation for an incompressible fluid simplifies considerably:

\[
-\vec{v} \text{grad} \rho + \text{div} (\vec{v} \rho) = 0 \tag{3.12}
\]

\[
\rightarrow -\vec{v} \text{grad} \rho + \vec{v} \text{grad} \rho + \rho \text{div} \vec{v} = 0
\]

\[
\rightarrow \quad \text{div} \vec{v} = 0 \tag{3.13}
\]

**Continuity equation for incompressible fluids**

- \(\vec{v}\) has no sources or sinks (as the magnetic field in electrodynamics)
- \(\vec{v}\) is called a solenoidal field and can be derived from a vector potential \(\vec{A}\):

\[
\vec{v}(\vec{r}, t) = \text{curl} \vec{A}(\vec{r}, t)
\]
3.2.5 Temperature

From the conservation of the total energy a non-linear diffusion equation for the temperature field of a fluid can be derived. It reads:

\[ \dot{T}(\vec{r}, t) + \vec{v}(\vec{r}, t) \text{grad} T(\vec{r}, t) = \kappa \Delta T(\vec{r}, t) \]

where \( \kappa \) is the thermal diffusivity. For details we refer to the literature (Landau, Lifschitz).

3.3 Hydrostatics

Hydrostatics describe that part of fluid dynamics, where fluids are not moving and are in mechanical equilibrium. Then the velocity and the time derivatives of all quantities must vanish:

\[ \vec{v} = 0, \quad \dot{\rho} = 0 \]

3.3.1 Basic equations

However, pressure and temperature inhomogeneities in space can occur. The Euler equations simplify to

\[ \text{grad} p = \vec{f} \]

- pressure balances the outer forces
- only possible if \( \text{curl} \vec{f} = 0 \) \( \Rightarrow \vec{f} = -\text{grad} U \)

\[ P + U = \text{const} \]

As an example we consider a (compressible) fluid or a gas in the constant gravitational field of the earth
For symmetry reasons, \( p = p(z) \). Then

\[
\frac{dp}{dz} = f_z = -g \, \rho(z) \tag{3.14}
\]

or

\[
p(z) = -g \int_{z_0}^{z} \rho(z) \, dz \tag{3.15}
\]

Thus we can compute the pressure distribution if we know the density \( \rho(z) \). But the density is in turn linked to the pressure by a material law.

In general we have

\[
\rho = \rho(p)
\]

### 3.3.2 Examples for pressure and density distribution

We consider the two special cases:

- Incompressible fluid:
  \[
  \rho = \rho_0 = \text{const}
  \]

- Ideal gas
  \[
  \rho = \frac{m_0}{RT} \rho
  \]
In both cases we can integrate

\[ \frac{dp}{dz} = -g \rho(p) \]

to

\[ \int_{p_0}^{p(z)} \frac{dp'}{\rho(p')} = -g(z - z_0) \]

with

\[ p_0 = p(z_0) \]

If the fluid is incompressible, we find

\[ p(z) - p_0 = -g \rho_0(z - z_0) \]

\[ p(z) = p_e - g \rho_0(z - h) \]

where we put \( z_0 = h \) and \( p_0 = p_e \), the pressure of the environment.

The case of an ideal gas yields the barometric formula and will be discussed in the exercises.
3.4. Potential Flows and Bernoulli’s Theorem

3.4.1 Potential flow

A flow is called “potential flow”, if it has no vortices:

\[
\text{curl} \, \vec{v} = 0 \tag{3.16}
\]

Then the flow is laminar, no shear flows are possible. It can be shown that if a perfect fluid once has no vortices, it will be free of vortices also in its future.

If (3.16) holds, \( \vec{v} \) has a potential \( \Psi(\vec{r}, t) \):

\[
\vec{v} = \text{grad} \, \Psi
\]

For incompressible potential flows one has in addition

\[
\text{div} \, \vec{v} = \text{div} \, \text{grad} \, \Psi = 0
\]

and therefore

\[
\Delta \Psi = 0 \tag{3.17}
\]

This is a Laplace equation which determines \( \Psi \) in a unique way, if suitable boundary conditions are provided.

The Euler equations for an incompressible potential flow \( \rho = \text{const} = \rho_0 \) take a much simpler form

\[
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho_0} \text{grad} \, p + \frac{1}{\rho_0} \vec{f}
\]

\[
= \frac{1}{2} \nabla \vec{v}^2 - \vec{v} \times (\text{curl} \, \vec{v}) - \nabla (p/\rho_0)
\]

\[
\frac{1}{2} \nabla (\nabla \Psi)^2 = 0
\]
This can be written as

\[ \nabla \cdot \left[ \dot{\Psi} + \frac{1}{2} (\nabla \Psi)^2 + \frac{p}{\rho_0} \right] = \frac{1}{\rho_0} \vec{f} \]  

(3.18)

The l.h.s. is a gradient of a scalar. This must be also true for the r.h.s. Then it follows that a potential flow can solve the Euler equations only for conservative force fields:

\[ \vec{f} = -\nabla U, \quad U : \text{potential energy} \]

Finally one can integrate (3.18) to:

\[ \dot{\Psi} + \frac{1}{2} (\nabla \Psi)^2 + \frac{p + U}{\rho_0} = \text{const} \]  

(3.19)

### 3.4.2 Receipt for solution of problems

- solve \( \Delta \Psi = 0 \) with b. c. → \( \Psi(\vec{r},t) \)
- Determine pressure from (3.19)

### 3.4.3 Example: Plane (2D) flow around a cylinder

Task: determine the laminar (vorticity free) flow and the pressure of an incompressible fluid around an infinitely long cylinder with radius \( R \).
Polar coordinates:

\[
\begin{align*}
  x &= r \cos \varphi, \\
  y &= r \sin \varphi \\
  \Psi &= \Psi(r, \varphi)
\end{align*}
\]

- solve \( \Delta \Psi = 0 \)

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) \Psi = 0
\]

Boundary conditions: The velocity normal to the cylinder has to vanish (impermeable wall):

\[
v_r = \left. \frac{\partial \Psi}{\partial r} \right|_{r=R} = 0
\]

Far from the cylinder, the fluid flows parallel to the x-axis, giving the b.c.:

\[
\Psi(r \to \infty) = V_\infty x = V_\infty r \cos \varphi
\]

As a solution one finds

\[
\Psi(r, \varphi) = V_\infty \cdot \left( r + \frac{R^2}{r} \right) \cos \varphi
\]

- Find pressure distribution from the Euler equation.

\[
p = c_0 - \frac{\rho_0}{2} (\nabla \Psi)^2
= p_\infty + \frac{\rho_\infty v_\infty^2 R^2}{r^2} \left( 2 \cos^2 \varphi - 1 - \frac{1}{2} \frac{R^2}{r^2} \right)
\]

### 3.4.4 Bernoulli’s theorem

The famous theorem of Bernoulli (1738) is a special case of eq. (3.19) for stationary or steady flows \( \dot{v} = 0 \), \( \rightarrow \Psi = 0 \):
CHAPTER 3. THE EULER EQUATIONS

\[ \frac{1}{2}v^2 + \frac{p + U}{\rho_0} = \text{const} \]  
(3.20)

This corresponds to the conservation of energy and is valid on each point in the flow.

As an application we derive the formula of Torricelli.

A fluid is in a vessel with height \( h \). Determine the velocity \( v \) of the fluid streaming through the valve on the bottom. Assume that the liquid inside the vessel is moving slowly and can be approximated being stationary.

From Bernoulli’s theorem we find

\[ p_1 + \rho gh = p_0 + \frac{1}{2} \rho v^2 \]

what can be solved for \( v \):

\[ v = \sqrt{\frac{2(p_1 - p_0)}{\rho} + 2gh} \]

If the external pressure does not depend on \( z \) this simplifies to the formula of Torricelli:

\[ v = \sqrt{2gh} \]
Chapter 4

The Navier-Stokes Equations

• the Navier-Stokes eqs. describe the motion of real fluids (liquid and gas) with *viscosity* under external pressure and forces.

• firstly stated as a model by C. L. Navier (1827)

• later derived systematically from continuum mechanics by G. G. Stokes (1845)

4.1 Stress tensor

• for ideal fluids, we found in the previous chapter

\[ T_{ij} = T_{ij}^E = -p \delta_{ij} \]

• for viscous fluids, we must add shear stress

\[ T_{NS} = T^E + \sigma \]

\( \sigma \): viscous stress tensor

\text{div} \( \sigma \): viscous forces, friction

\[ \rightsquigarrow \] To arrive at the Navier-Stokes eqs., we have to add \text{div} \( \sigma \) on the R.H.S. of the Euler equations

How to find \( \sigma \)?
Newton: if a fluid moves homogeneously, there can be no viscous stress.

\[ \mathbf{\sigma} = \mathbf{\sigma}(\partial_i v_j) \]

for most fluids, \( \mathbf{\sigma} \) depends linearly on \( \partial_i v_j \)

- there are exceptions, called Non-Newtonian-Fluids, like toothpaste, mayonnaise, blood, colors (plastic fluids), pseudo-plastic fluids, etc.

Here we shall consider only **Newtonian fluids**

There are three different possibilities to construct a tensor \( \sim \partial_i v_j \) if the system is isotropic. The general form reads

\[ \mathbf{\sigma} = a (\nabla \circ \mathbf{v}) + b (\mathbf{v} \circ \nabla) + c \sum_{\ell} \partial_{\ell} v_{\ell} \]

or in components

\[ \sigma_{ij} = a \partial_i v_j + b \partial_j v_i + c \delta_{ij} \sum_{\ell} \partial_\ell v_\ell \]  \hspace{1cm} (4.1)

\( a, b, c \) are material constants and related to viscosity.

In fact, only two of the three constants are independent. This can be seen considering a uniformly rotating fluid around the \( z \)-axis. In such a situation, the fluid is at rest in the rotating frame of references and no friction occurs.

Inserting

\[ \mathbf{v} = \mathbf{\omega} \times \mathbf{r} \]

into (4.1) gives after a little algebra (try it in components and use the \( \varepsilon \)-tensor) the condition

\[ a = b \]

for \( \mathbf{\sigma} = 0 \)

Inserting this into (4.1) yields for the viscous stress tensor

\[ \mathbf{\sigma} = 2a \mathbf{D} + c \text{ tr}(\mathbf{D}) \cdot \mathbf{1} \]

with
4.2 Viscosities

Usually, the viscous stress tensor is written in another form. After some manipulations, one can decompose it in a trace free part (pure shearing, no volume change) and a diagonal part (only volume changes):

\[
\sigma = 2a D + c \text{tr}(D) \cdot 1
\]

\[
= 2a D - \frac{2a}{3} \text{tr}(D) \cdot 1 + \frac{2a}{3} \text{tr}(D) \cdot 1 + c \text{tr}(D) \cdot 1
\]

\[
= 2 \eta, 1. \text{viscosity} \left( D - \frac{1}{3} \text{tr}(D) \cdot 1 \right) + \frac{2a}{3} + c \text{tr}(D) \cdot 1 \quad \text{(4.5)}
\]

\[
\sigma = 2\eta \left( D - \frac{1}{3} \text{tr}(D) \cdot 1 \right) + \zeta \text{tr}D \cdot 1
\]

or in components

\[
\sigma_{ij} = 2\eta \left( D_{ij} - \frac{1}{3} \sum \delta_{i\ell} \cdot D_{\ell \ell} \cdot \delta_{\ell j} \right) + \zeta \sum D_{\ell \ell} \cdot \delta_{ij}
\]

\[
\text{viscosity, 1st viscosity } \eta
\]

\eta: \text{(dynamic) viscosity, } \nu = \eta / \rho: \text{kinematic viscosity}

shear stress = \eta \cdot \text{share rate} (\eta, \nu \neq \text{shear modulus})
2nd viscosity $\zeta$

\[
\text{tr } \sigma = 3\zeta \text{tr } D = 3\zeta \text{ div } \vec{v} = 3\zeta \dot{\theta}, \quad (4.8)
\]

$\theta$: volume dilatation, $\dot{\theta}$: rate of dilatation

\[
\text{dilatation stress } = \zeta \cdot \text{dilatation rate } (\zeta \doteq \text{ bulk modulus})
\]
4.3. THE NAVIER-STOKES EQUATIONS

<table>
<thead>
<tr>
<th>material</th>
<th>dyn. v. (η)</th>
<th>kin. v. (ν)</th>
<th>material</th>
<th>dyn. v. (η)</th>
<th>kin. v. (ν)</th>
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<tbody>
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<td>water</td>
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<td>1.0</td>
<td>petroleum</td>
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<td>0.76</td>
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<tr>
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<td>0.11</td>
<td>pentane</td>
<td>0.23</td>
<td>0.37</td>
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<td>alcohol</td>
<td>1.8</td>
<td>2.2</td>
<td>blood (37°)</td>
<td>4 - 25</td>
<td>4 - 25</td>
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<tr>
<td>glycerine</td>
<td>850</td>
<td>650</td>
<td>honey</td>
<td>10.000</td>
<td>7.000</td>
</tr>
<tr>
<td>glass</td>
<td>10^{23}</td>
<td>0.4 · 10^{23}</td>
<td>melted polymers</td>
<td>10^{3} − 10^{6}</td>
<td>10^{3} − 10^{6}</td>
</tr>
<tr>
<td>oxygen</td>
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<td>13</td>
<td>argon</td>
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<td>105</td>
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<tr>
<td>hydrogen</td>
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<td>90</td>
<td>grape juice</td>
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<td>0.017</td>
<td>14</td>
<td>air</td>
<td>0.018</td>
<td>14</td>
</tr>
</tbody>
</table>

Dynamic viscosities in mPa·s or 10⁻³ kg/ms, kinematic viscosities in 10⁻⁶ m²/s or centi-stokes. All values at room temperature.

4.3 The Navier-Stokes equations

The total balance of forces, including viscosities, now reads

\[ \rho \frac{d\vec{v}}{dt} = \text{div} \mathbf{T} + \vec{f} = -\text{grad} p + \text{div} \mathbf{\sigma} + \vec{f} \] (4.9)

Now we compute \( \text{div} \mathbf{\sigma} \):

\[ (\text{div} \mathbf{\sigma})_j = \partial_j \sigma_{ij} = \eta \Delta v_j + (\zeta + \frac{1}{3} \eta) \partial_j \text{div} \vec{v} \] (4.10)

Assume \( \eta, \zeta \) to be constant in space (this must not be the case if the fluid is not isothermal or consists of a spatially inhomogeneous mixture of different components):

\[ \rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = -\text{grad} p + \vec{f} + \eta \Delta \vec{v} + (\zeta + \frac{1}{3} \eta) \text{grad} \text{div} \vec{v} \] (4.11)

These are the basic equations for compressible (pure, isothermal) fluids or gases. They have to be completed by material laws. The subject of compressible liquids is sometimes called “aerodynamics”.

For incompressible fluids (usually liquids or gases under high pressure), they simplify to:

\[
\rho_0 \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right) = -\nabla p + \vec{f} + \eta \Delta \vec{v} \\
\text{div} \vec{v} = 0
\]  

(4.12)

The subject of incompressible fluids is called “hydrodynamics”.

4.4 Boundary conditions

1. No slip conditions.
   - rigid, impermeable boundaries

   \[
   \vec{v} = 0 \text{ at the walls}
   \]

   Incompressible fluids: (wall at \( z = 0 \))

   \[
   \text{div} \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \\
   v_z = 0 \quad \text{and} \quad \frac{\partial v_z}{\partial z} = 0
   \]  

   (4.13)

In a fluid without friction (Euler eq.) there is only one condition:

Impermeable wall (no flux)
4.4. BOUNDARY CONDITIONS

Reason: Navier-Stokes eq. includes higher derivatives ($\nu \Delta \vec{v}$)

- The approximation

\[
\text{Navier-Stokes} \quad \rightarrow \quad \text{Euler}
\]

\[
\nu \quad \rightarrow \quad 0
\]

is not systematic and generally not valid!

2. Free surface condition (no flux)

viscous fluids and Euler fluids:

\[
\nu_z = 0
\]

In viscous fluids, the total stress along the surface must balance.

1. constant surface tension
\[ \hat{t}_i \cdot \hat{t}_i = \hat{n} \cdot \sigma \cdot \hat{n} = 0 \]

\[ \hat{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \hat{i}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{i}_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \] (4.14)

\[ \rightarrow \quad \sigma_{xz} = 0, \quad \sigma_{yz} = 0 \quad \text{at the free surface} \]

incompressible fluids

\[ \frac{\partial}{\partial z} \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0 \] (4.15)

\[ \frac{\partial}{\partial x} \frac{\partial v_x}{\partial z} + \frac{\partial}{\partial y} \frac{\partial v_y}{\partial z} + \frac{\partial^2 v_z}{\partial z^2} = 0 \quad |z = 0 \] (4.16)

\[ \frac{\partial^2 v_z}{\partial z^2} = 0 \]

2. non constant surface tension (e.g. due to temperature variations)

\[ \vec{f}_s \]

\[ \text{cold} \quad \text{hot} \]

\[ \text{surface , } \gamma = \text{surface tension} \]

force (stress) at surface: \( \vec{f}_s = \nabla \gamma \)

\[ \hat{n} \cdot \sigma \cdot \hat{n} = \hat{i}_i \cdot \nabla \gamma, \quad \hat{i}_i = \hat{i}_x, \hat{i}_y \]

\[ \eta \frac{\partial v_x}{\partial z} = \frac{\partial \gamma}{\partial x}, \quad \eta \frac{\partial v_y}{\partial z} = \frac{\partial \gamma}{\partial y} \] (4.17)

or (with continuity eq.)

\[ \eta \frac{\partial^2 v_z}{\partial z^2} = -\Delta_2 \gamma \]
4.4. BOUNDARY CONDITIONS

4.4.1 Application

Compute the stationary flow in an open layer which is laterally heated. Assume a horizontal pressure difference $\delta p$ and a horizontal temperature difference $\delta T$. Which $\delta T$ is needed to keep the fluid in rest at the free surface $z = h$?

\[ \vec{v} = \begin{pmatrix} v_x(z) \\ 0 \\ 0 \end{pmatrix}, \quad \text{div} \vec{v} = 0 \quad \text{is fulfilled} \quad (4.18) \]

Navier-Stokes eqs. (for a stationary flow all time derivatives have to vanish):

\[ \rho \left( \dot{v}_x + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = \eta \Delta v_x - \frac{\delta p}{L} \quad (4.19) \]

\[ - \eta \frac{\partial^2 v_x}{\partial z^2} - \frac{\delta p}{L} = 0 \quad \implies \quad v_x(z) = a + bz + \frac{1}{2\eta} \frac{\delta p}{L} z^2 \]

The coefficients $a, b$ can be computed from the boundary conditions

\[ v_x(0) = 0 \quad \implies \quad a = 0 \]

\[ \left. \frac{dv_x}{dz} \right|_h = \frac{1}{\eta} \frac{d\gamma}{dx} \quad (4.20) \]

assume that the surface tension is a linear function of temperature:

\[ \gamma = c_0 - c_T (T - T_0), \quad c_T = - \left( \frac{\partial \gamma}{\partial T} \right)_{T_0} \quad (4.21) \]

\[ \frac{d\gamma}{dx} = -c_T \frac{dT}{dx} = -c_T \frac{\delta T}{L} \quad (4.22) \]

Inserting this into (4.20) leads to
\[ b + \frac{1}{\eta} \frac{\delta p}{L} = \frac{c_T}{\eta} \frac{\delta T}{L} \]

\[ b = \frac{1}{\eta L} (h \cdot \delta p + c_T \cdot \delta T) \]

\[ v_x(z) = \frac{1}{\eta L} \left[ \frac{1}{2} z^2 \delta p - (h \cdot \delta p + c_T \delta T) z \right] \quad (4.23) \]

The critical temperature where the fluid is in rest at the surface follows from:

\[ v_x(z = h) = 0 \]

\[ \delta T = \frac{1}{2} \frac{h}{c_T} \delta p \quad (4.24) \]

for \( \delta p = 0 \) we have a pure shear flow

\[ v_x = \frac{-c_T \delta T}{\eta L} z \quad (4.25) \]
4.5 Stream functions

We consider incompressible flows \( \text{div} \vec{v} = 0 \).

4.5.1 Plane flows

Plane flow (two-dimensional):

\[
\begin{align*}
  v_x &= v_x(x, y), \quad v_y = v_y(x, y), \quad v_z = 0
\end{align*}
\]

The stream function is a function whose contour lines are equal to the stream lines of the flow (at a fixed moment). The stream lines are computed by

\[
\begin{align*}
  dx &= v_x dt, \quad dy = v_y dt
\end{align*}
\]

or, eliminating \( dt \):

\[
\begin{align*}
  v_x dy - v_y dx &= 0 \quad (4.26)
\end{align*}
\]

The total differential of a function \( \Psi(x, y) \) reads

\[
d\Psi = \frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} dy
\]

If the stream lines are contour lines of \( \Psi \), \( d\Psi = 0 \) along the streamlines. Comparing with (4.26) yields

\[
\begin{align*}
  v_x &= f(x, y) \frac{\partial \Psi}{\partial y}, \quad v_y = -f(x, y) \frac{\partial \Psi}{\partial x}
\end{align*}
\]

with an arbitrary function \( f \). This can be determined by the continuity equation

\[
\text{div} \vec{v} = \frac{\partial}{\partial x} \left( f \frac{\partial \Psi}{\partial y} \right) - \frac{\partial}{\partial y} \left( f \frac{\partial \Psi}{\partial x} \right) = 0
\]

Evaluating the derivatives gives

\[
\frac{\partial f}{\partial x} \frac{\partial \Psi}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial \Psi}{\partial x} = 0
\]

For arbitrary \( \Psi \) this can only be solved if

\[
\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \text{or} \quad f = \text{const}
\]

Thus we can put \( f = 1 \) and find

\[
\begin{align*}
  v_x &= \frac{\partial \Psi}{\partial y} \\
  v_y &= -\frac{\partial \Psi}{\partial x}
\end{align*}
\]
4.5.2 Axisymmetric flows

The same procedure can be applied for a flow which is symmetric with respect to a certain axis. We put the $z$-axis equal to that symmetry axis. Then the flow is best described using cylindrical or spherical coordinates.

**Cylindrical coordinates**

One has

$$v_r = v_r(r, z), \quad v_z = v_z(r, z), \quad v_\phi = 0$$

From

$$v_r dz - v_z dr = 0 \quad (4.27)$$

one finds

$$v_r = f(r, z) \frac{\partial \Psi}{\partial z}, \quad v_z = -f(r, z) \frac{\partial \Psi}{\partial r}$$

Inserting this into the continuity equation gives the two conditions for $f$

$$f + r \frac{\partial f}{\partial r} = 0, \quad \frac{\partial f}{\partial z} = 0$$

which is solved by

$$f = f(r) = \frac{1}{r}$$

Finally

$$\begin{cases} v_r = \frac{1}{r} \frac{\partial \Psi}{\partial z} \\ v_z = -\frac{1}{r} \frac{\partial \Psi}{\partial r} \end{cases}$$

**Spherical coordinates**

Starting from

$$v_r = v_r(r, \theta), \quad v_\theta = v_\theta(r, \theta), \quad v_\phi = 0$$

the stream lines are now computed by

$$dr = v_r dt, \quad r d\theta = v_\theta dt$$
and eliminating $dt$:

$$v_r d\theta - \frac{v_\theta}{r} dr = 0$$

(4.28)

One has

$$v_r = f(r, \theta) \frac{\partial \Psi}{\partial \theta}, \quad v_\theta = -r f(r, \theta) \frac{\partial \Psi}{\partial r}$$

Inserting this into the continuity equation gives the two conditions for $f$

$$f \cot \theta + \frac{\partial f}{\partial \theta} = 0, \quad 2f + r \frac{\partial f}{\partial r} = 0$$

This can be solved by a product

$$f(r, \theta) = h(r) \cdot g(\theta)$$

and gives

$$h(r) = \frac{1}{r^2}, \quad g(\theta) = \frac{1}{\sin \theta}$$

and

$$\begin{align*}
  v_r &= \frac{1}{r^2 \sin \theta} \frac{\partial \Psi}{\partial \theta} \\
  v_\theta &= -\frac{1}{r \sin \theta} \frac{\partial \Psi}{\partial r}
\end{align*}$$

### 4.6 Viscous potential flows

if

$$\text{curl} \vec{v} = 0 \rightarrow \vec{v} = \nabla \Phi$$

with the velocity potential

$$\Phi(\vec{r}, t)$$

$\vec{v}$ is called a “potential flow”. For incompressible fluids we have

$$\text{div} \vec{v} = \text{div} (\nabla \Phi) = \Delta \Phi = 0$$

Navier-Stokes eq. for a potential incompressible flow:
4.7 Boundary layers

Assume a vortex free, potential flow (incompressible)

\[ \vec{v} = \nabla \Phi, \quad \Delta \Phi = 0 \]

Now we need boundary conditions for \( \Phi \) to solve the Laplace equation. There are two possibilities:

1st boundary value problem (Dirichlet) \quad or \quad 2nd boundary value problem (Neumann)

\[ \Delta \Phi = 0 \quad \Phi (\vec{r}) \quad \Delta \Phi = 0 \]

\[ \Phi \big|_\Omega \quad \text{given} \quad \frac{\partial \Phi}{\partial n} \big|_\Omega \quad \text{given} \]
Now consider a viscous fluid on an impermeable no-slip wall (at $\Omega$):

$$v_n|_\Omega = \frac{\partial}{\partial n} \Phi|_\Omega = 0, \quad \Rightarrow \quad \frac{\partial}{\partial n} \Phi|_\Omega = 0$$

and

$$v_t|_\Omega = \frac{\partial}{\partial t} \Phi|_\Omega = 0, \quad \Rightarrow \quad \Phi|_\Omega = \text{const}$$

So we have both boundary value problems. But than the system is over determined!

solution: concept of boundary layers

Ludwig Prandtl – Boundary layer theory:

at least close to a wall (or an obstacle) the fluid cannot be potential $\rightarrow$ vortices and shear flow occur there.

\[ \vec{\omega} = \frac{1}{2} \text{curl} \vec{v} \]

one can derive an equation for the vorticity $\vec{\omega} = \frac{1}{2} \text{curl} \vec{v}$ of the form

\[ \frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla) \vec{v} + \nu \Delta \vec{\omega} \]  \hspace{1cm} (4.30)
$v = \nabla \Phi$

laminar flow

Diffusion

$U_0$

transport

Karman Vortex Street

Diffusion

$\mathbf{v} \neq \nabla \Phi$

Turbulence

$R_e \sim U_0$, Reynolds number

$R_e$ characterizes the flow

$R_e < R_{crit}^*$: laminar, boundary layer located at obstacle

$R_e > R_{crit}^*$: turbulent, separation of boundary layer
Part II

Waves
Chapter 5

Sound Waves

5.1 Preliminaries

Sound waves exist in solids, liquids and gases. To allow for propagation of sound waves, the medium must be compressible.

An incompressible fluid behaves like a rigid body. The body moves without deformation and oscillations on one side are transmitted instantaneously to the other (arbitrarily distant) side:

\[ \text{rigid body} \quad \text{instantaneous propagation} \]

this corresponds to an infinite sound speed!

A compressible fluid behaves like an elastic solid. Now oscillations propagate through the solid in form of compression waves. Their speed is finite and depends on the elastic properties, pressure, temperature etc.

\[ \text{elastic body} \quad \text{finite velocity} \]
If the amplitude of these compression waves is (infinitesimally) small, they are called “acoustic waves” or “sound waves”.

### 5.2 Sound speed

We consider a front that separates two regions with different pressure and density. The front moves to the left with constant velocity $c$. The fluid left from the front is in rest.

In the co-moving frame (with $c$ to the left) one has a stationary front.

Now we take a finite volume around the front with surface $A$.
Conservation of mass yields

\[ c \rho_0 A = (c + du) \left( \rho_0 + d\rho \right) A \approx c \rho_0 A + \rho_0 A \, du + c A \, d\rho \]

or

\[ du = -\frac{c}{\rho_0} \, d\rho \]  \hspace{1cm} (5.1)

If \( d\rho > 0 \) (compression wave), the fluid behind the front moves with \( du < 0 \) (in the direction of the front motion).

Now we take the conservation of the momentum (no friction, Euler eq.):

\[ p_0 A - (p_0 + d p) A = A \cdot \rho_0 \cdot c \cdot (c + du) - A \cdot \rho_0 \cdot c \]

or

\[ dp = -\rho_0 \cdot c \cdot du \]  \hspace{1cm} (5.2)

Eliminating \( du \) from (5.1) and (5.2) one finds the important result

\[ c^2 = \frac{dp}{d\rho} \]

To compute the speed of sound one needs a state equation \( p = p(\rho) \).

### 5.3 Wave equation for sound waves

For small amplitude waves, viscosity and nonlinearities can be neglected. The linearized Euler eq. and continuity eq. read

\[ \rho \frac{\partial \vec{v}}{\partial t} = -\text{grad} \, p \]  \hspace{1cm} (5.3)

\[ \frac{\partial \rho}{\partial t} = -\text{div} \, (\vec{v} \, \rho) \]  \hspace{1cm} (5.4)
5.3.1 Compression waves

We use the decomposition

\[ \rho \vec{v} = \rho \vec{v}_1 + \rho \vec{v}_2 \]

with

\[ \text{curl} \, \rho \vec{v}_1 = 0 \quad \Rightarrow \quad \rho \vec{v}_1 = \text{grad} \Phi \]

and

\[ \text{div} \, \rho \vec{v}_2 = 0 \quad \Rightarrow \quad \rho \vec{v}_2 = \text{curl} \vec{A} \]

The first part describes a pure compression without shearing or vortices. The second part corresponds to a shearing without volume change.

Inserting this into (5.3) yields

\[ \text{grad} \dot{\Phi} + \text{curl} \dot{\vec{A}} = -\text{grad} p \]

and

\[ \text{curl} \dot{\vec{A}} = 0 \]

The vortices remain constant and are conserved. Then we can integrate (5.5) to

\[ \dot{\Phi} = p_0 - p \]

with a certain constant \( p_0 \). Inserting the decomposition into (5.4) one gets

\[ \dot{\rho} = -\text{div} \text{grad} \Phi - \text{div} \text{curl} \vec{A} = -\Delta \Phi \]

Again we need a state equation of the form \( p = p(\rho) \). Then we can differentiate (5.6) with respect to time and use (5.7)

\[ \ddot{\Phi} = -\dot{p} = -\frac{dp}{d\rho} \dot{\rho} = \frac{dp}{d\rho} \Delta \Phi \]

or

\[ \dot{\Phi} - c^2 \Delta \Phi = 0 \]
5.3. **WAVE EQUATION FOR SOUND WAVES**

This is a wave equation for sound waves with phase speed \( c^2 = dp/d\rho \). From (5.6) the pressure waves can be computed.

### 5.3.2 State equation

To evaluate \( c \), Isaac Newton used the state equation of a perfect gas:

\[
p = \rho R T
\]

(5.8)

For air at room temperature this gives \( c \approx 290 \text{ m/s} \), compared to \( c = 340 \text{ m/s} \) from the experiment. Newton assumed “unclean air” being the reason for the large discrepancy.

About 100 years later, Laplace showed that the compression is not isothermal but adiabatic (or isentrop). The temperature changes during compression periodically. But the motion is so fast, that the temperature fluctuations are not transported to the environment by heat flux.

For an adiabatic process, the relation between pressure and density reads

\[
p = \text{const} \cdot \rho^\gamma
\]

where \( \gamma = c_p/c_V \) is the adiabatic exponent and \( c_p, c_V \) is the specific heat under constant pressure and volume, respectively. For a mono-atomic perfect gas one has \( \gamma = 5/3 \), for a di-atomic gas \( \gamma = 7/5 \).

Thus

\[
\frac{dp}{d\rho} = \gamma \cdot \text{const} \cdot \rho^{\gamma - 1}
\]

Using (5.8) one determines the constant to \( RT \rho^{1-\gamma} \) and finally finds

\[
c = \sqrt{\gamma RT},
\]

a value, which is in excellent agreement with the experiment.
Chapter 6

Surface Waves

6.1 Preliminaries

We consider waves on the surface of a liquid layer (river, lake, ocean)

\[ P_0: \text{external pressure} \]
\[ \rho: \text{density of fluid} \]

- find equation for \( h(x, y, t) \)
- find the internal motion of the fluid \( \vec{v} \)
- can instabilities occur? \( \rightarrow \) (Part III)

Assumptions and approximations:

- viscosity is not important \( \rightarrow \) Euler equations
- no vorticity, \( \text{curl} \vec{v} = 0 \) \( \rightarrow \) Potential flow
- incompressible fluid, \( \text{div} \vec{v} = 0 \)
CHAPTER 6. SURFACE WAVES

- small surface deflection, \[ \frac{|h - h_0|}{h_0} \ll 1 \]

6.2 Gravity waves

6.2.1 equations for flow

Euler equations \((\rho = \text{const})\)

\[ \dot{\vec{v}} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho} \text{grad} (P + U) \]

with (Potential flow)

\[ \vec{v} = \nabla \Phi \]

and the formula

\[ (\vec{v} \cdot \nabla)\vec{v} = \frac{1}{2} \frac{\nabla v^2}{\nabla (\nabla \Phi)^2} - \nabla \times (\nabla \times \vec{v}) = 0 \]

We find

\[ \nabla \left[ \Phi + \frac{1}{2} (\nabla \Phi)^2 \right] = -\nabla \left[ \frac{P + U}{\rho} \right] \quad (6.1) \]

or, after integration

\[ \Phi = -\frac{P + U}{\rho} - \frac{(\nabla \Phi)^2}{2} \]

\[ \Delta \Phi = 0 \quad (6.2) \]

This are the basic equation for an incompressible, vortex free fluid (cmp. part I, chapt. 3.4)

The water in a constant gravitation field has the potential energy:

\[ U = \rho g z + U_0 \]

6.2.2 Equation for the location of the free surface

Let the surface be located at \( z = h(x, y, t) \)
6.2. GRAVITY WAVES

If there is a vertical velocity component, the surface moves with that velocity: 
\[ h = v_z(z = h) \]

but also a horizontal velocity takes the surface with it, according to:

\[ \dot{h} = -v_x(z = h) \partial_x h \]

both together yields (in three dimensions)

\[ \dot{h} = -v_x|_h \partial_x h - v_y|_h \partial_y h + v_z|_h \]

or, using the potential:

\[ \dot{h} = -\nabla^2 \Phi|_h \cdot \nabla h + \partial_z \Phi|_h \]

Now we evaluate eq. (6.2) at the surface \( z = h \):

\[ \Phi|_h = -\frac{g(h - h_0)}{U(z=h)} - \frac{1}{2} (\nabla \Phi|_h)^2 \]

where we used

\[ U = \rho g z + U_0, \quad \text{with} \quad U_0 = P_0 - \rho g h_0 \]
6.2.3 Basic equations and linear solutions

\[ \Delta \Phi = 0 \quad (6.6) \]
\[ \Phi_h = -g(h - h_0) - \frac{1}{2} (\nabla \Phi)_h^2 \quad (6.7) \]
\[ h = -\nabla \Phi|_h \cdot \nabla h + \partial_z \Phi|_h \quad (6.8) \]

Now we assume that the basic state is that of a flat surface \( h = h_0 \) where the fluid is in rest, \( \Phi = 0 \) (hydrostatic solution)

Consider small deviations from that state

\[ \eta(x, t) = h(x, t) - h_0, \quad \Phi(x, t) \quad (6.9) \]

eqs. (6.7), (6.8) can be linearized:

\[ \Phi|_z = -g \eta \]
\[ \dot{\eta} = \partial_z \Phi|_h \]
\[ \Rightarrow \partial_z \Phi + \frac{1}{g} \Phi = 0 \quad (6.10) \]

We assume a solution in form of waves:

\[ \Phi = \xi(t) \cdot f(z) e^{ikx} \quad (6.11) \]

inserting this into (6.6) yields

\[ f'' - k^2 f = 0 \quad \rightarrow \quad f(z) \sim e^{\pm k|z-h_0|} \]

and with the boundary condition (infinitely deep layer)

\[ \Phi(z \to -\infty) = 0 \quad \rightarrow \quad f(z) \sim e^{\pm k|z-h_0|} \quad (6.12) \]

Substitute (6.11), (6.12) into (6.10) gives

\[ |k| \dot{\xi} + \frac{1}{g} \xi = 0 \quad \rightarrow \quad \xi(t) = e^{\pm i\omega t}, \quad (6.13) \]
6.2. GRAVITY WAVES

the equation of an harmonic oscillator with the frequency

$$\omega = \sqrt{|k|g}$$

Thus we have as a solution of the linearized problem

$$\Phi = Ae^{|k|(z-h_0)} \cos(kx \pm \omega t)$$

$$h = h_0 \pm A \sqrt{\frac{k}{g}} \sin(kx \pm \omega t)$$

(6.14) (6.15)

and from there the velocity components

$$v_x = \partial_x \Phi = -kAe^{|k|(z-h_0)} \sin(kx \pm \omega t)$$

$$v_z = \partial_z \Phi = |k|Ae^{|k|(z-h_0)} \cos(kx \pm \omega t)$$

(6.16) (6.17)

This corresponds to traveling waves with the phase velocity

$$c = \frac{\omega}{k}$$

The dispersion relation reads

$$\omega = \sqrt{kg}$$

Using this, the phase velocity can be expressed as

$$c = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$$

– the longer the wave length, the faster the wave propagates

How do the trajectories of volume element (its path) look?

To answer this, one has to solve the system

$$\frac{dx}{dt} = v_x = -kAe^{|k|(z-h_0)} \sin(kx \pm \omega t)$$

$$\frac{dz}{dt} = v_z = |k|Ae^{|k|(z-h_0)} \cos(kx \pm \omega t)$$

(6.18) (6.19)

two coupled nonlinear ODE’s which can be solved only numerically.
Approximation: $|\vec{v}| \ll c$

With the initial condition $x_0 = x(t = 0), z_0 = z(t = 0)$ one can integrate

\[ x(t) = x_0 + \int_0^t v_x(x_0, z_0, t') dt' = x_0 + a(\cos(kx_0 - \omega t) - \cos kx_0) \quad (6.20) \]

\[ z(t) = z_0 + \int_0^t v_z(x_0, z_0, t') dt' = z_0 - b(\sin(kx_0 - \omega t) - \sin kx_0) \quad (6.21) \]

with

\[ a = -\frac{A k}{\omega} e^{k|z_0 - h_0|} \quad (6.22) \]

\[ b = \frac{A |k|}{\omega} e^{k|z_0 - h_0|} \quad (6.23) \]

- volume elements travel on circles with radius $|a| = |b| \sim e^{kr_0}$
- in time average, particles don’t travel at all!

But: nonlinear corrections leads to the so-called “Stokes drift”, an average velocity

\[ |\vec{v}_s| \sim a^2 \]

and parallel to $\vec{k}$. 

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6.3 The Shallow Water equations

Now we consider surface deformations in form of long waves. This does not mean only harmonic waves but can be any other form. It is important that the dimension (extension) in horizontal direction is large compared to the depth of the fluid.

Example of a "long wave"

\( \delta = \frac{h_0}{\ell} \ll 1 \)

Examples for "long waves" are:

- ocean waves near the shore
- Tsunamis
- Waves on a canal

To arrive at a dimensionless formulation of the problem, the variables of eqs. (6.6), (6.7), (6.8) are scaled in the following way:

\[
x = \tilde{x} \cdot \ell, \quad z = \tilde{z} \cdot h_0, \quad h = \tilde{h} \cdot h_0
\]

\[
t = \tilde{t} \cdot \tau \quad \Phi = \tilde{\Phi} \cdot \frac{\ell^2}{\tau}
\]

Then eqs. (6.6), (6.7), (6.8) read

\[
0 = \delta^2 \tilde{z} \tilde{\Phi} + \delta^2 \partial_{\tilde{x}}^2 \tilde{\Phi}
\]

\[
\dot{\tilde{\Phi}} = -G (\tilde{h} - 1) - \frac{1}{2} (\partial_{\tilde{x}} \tilde{\Phi})^2 - \frac{1}{2 \delta^2} (\partial_{\tilde{z}} \tilde{\Phi})^2
\]

\[
\delta^2 \dot{\tilde{h}} = -\delta^2 \partial_{\tilde{x}} \tilde{\Phi} \cdot \partial_{\tilde{x}} \tilde{h} + \partial_{\tilde{z}} \tilde{\Phi}
\]

(from here, we suppress the tildes). The non-dimensional number

\[
G = \frac{g \cdot h_0 \cdot \tau^2}{\ell^2}
\]
is called “gravitation number”.

Trick: we solve (6.26) by iteration (systematic perturbation analysis with respect to small $\delta$):

$$\Phi = \Phi^{(0)} + \delta^2 \Phi^{(2)} + \delta^4 \Phi^{(4)} + \ldots$$

(6.29)

this inserted into (6.26) gives:

$$\partial^2_{zz} \Phi^{(0)} + \delta^2 \left( \partial^2_{zz} \Phi^{(2)} + \partial^2_{xx} \Phi^{(0)} \right) + \delta^4 \left( \partial^2_{zz} \Phi^{(4)} + \partial^2_{xx} \Phi^{(2)} \right) + \ldots = 0$$

(6.30)

Since $\delta$ can be arbitrary, terms with the same order of $\delta$ must vanish:

- order $\delta^0$

$$\partial^2_{zz} \Phi^{(0)} = 0 \quad \rightarrow \quad \Phi^{(0)} = f_1(x,t) + f_2(x,t) \cdot z \quad (6.31)$$

with the boundary condition on the ground ($z = 0$)

$$v_z(z = 0) = \frac{\partial \Phi}{\partial z} \bigg|_{z=0} = f_2 = 0 \quad \Rightarrow \quad f_2 = 0 \quad (6.32)$$

and finally the important result

$$\Phi^{(0)} = \Phi^{(0)}(x,t) \quad (6.33)$$

- order $\delta^2$:

$$\partial^2_{zz} \Phi^{(2)} = -\partial^2_{xx} \Phi^{(0)} = -\Phi^{(0)''}$$

(6.34)

$$\rightarrow \quad \Phi^{(2)}(x,z,t) = -\Phi^{(0)''} \cdot \frac{z^2}{2} + \underbrace{f_3(x,t)}_{=0 \text{ (b. c)}} \cdot z + f_4(x,t) \quad (6.35)$$

- order $\delta^4$

- in the same way …

We write down the result up to the order $\delta^4$: 
\[ \Phi(x,z,t) = \Phi^{(0)}(x,t) + \delta^2 \left[ -\Phi^{(0)'''} \cdot \frac{z^2}{2} + f_4(x,t) \right] \]
\[ + \delta^4 \left[ \Phi^{(0)''} \cdot \frac{z^4}{24} - f''_4 \cdot \frac{z^2}{2} + f_6(x,t) \right] + \ldots \] (6.36)

- we know the z-dependence of \( \Phi \) explicitly !!
- if \( \Phi^{(0)}(x,t) \) is known, \( \Phi(x,z,t) \) can be determined.

Now we insert this into (6.27), (6.28) and take the lowest, non-trivial order:

\[ \dot{\Phi}^{(0)} = -G(h-1) - \frac{1}{2} (\partial_x \Phi^{(0)})^2 \] (6.37)
\[ \dot{h} = -\partial_x \Phi^{(0)} \cdot \partial_x h - h \cdot \partial_{xx} \Phi^{(0)} \] (6.38)

or in two horizontal dimensions \((x,y)\)

\[ \dot{\Phi} = -G(h-1) - \frac{1}{2} (\nabla^2 \Phi)^2 \] (6.39)
\[ \dot{h} = -\nabla^2 \Phi \cdot \nabla^2 h - h \cdot \Delta_2 \Phi \] (6.40)

These are the Shallow Water equations.

Advantage: only two equations instead of three

Big advantage: one spatial dimension is eliminated!

\[ 3D \rightarrow 2D \]
\[ 2D \rightarrow 1D \]

### 6.3.1 The linearized Shallow Water equations

We consider small deviations \( \eta \) from the constant depth \( h_0 = 1 \):
\( \eta = h - 1 \)

Then \( \Phi \) is also small and (6.37), (6.38) or (6.39), (6.40) can be linearized:

\[
\begin{align*}
\Phi &= -G\eta \\
\eta &= -\Delta_2 \Phi
\end{align*}
\] (6.41) (6.42)

Differentiating (6.42) with respect to time and eliminating \( \Phi \) yields a wave equation for \( \eta \)

\[
\ddot{\eta} - c^2 \Delta_2 \eta = 0
\]

with the phase velocity

\[ c = \sqrt{G} \]

rescaling all variables gives the velocity in dimensional form

\[ c = \sqrt{gh_0} \]

- phase velocity of long waves is constant!
- it depends only on the depth of the layer
6.3.2 Numerical solutions of the nonlinear Shallow Water equations

time evolution in 1D

snapshot in 2D
6.3.3 Shallow water waves on a modulated ground

boundary condition on the ground:

\[ \hat{n} \cdot \vec{v} = \hat{n} \cdot \nabla \Phi = 0 \]

Deriving the Shallow Water equations in the same manner as above, this gives rise to two new terms (underlined)

\[
\begin{align*}
\Phi &= -G(h - 1) - \frac{1}{2} (\nabla^2 \Phi)^2 \\
h &= - (\nabla^2 h) \cdot (\nabla^2 \Phi) - h \Delta_2 \Phi + (\nabla_2 f) \cdot (\nabla^2 \Phi) + f \Delta_2 \Phi
\end{align*}
\]

(6.43)

(6.44)

linearizing again yields a wave equation, now of the form

\[
\ddot{h} - G \nabla_2 [H(x,y) \nabla^2 h] = 0
\]

(6.45)

with \( H = 1 - f \) denoting the real depth of the flat water. If we neglect \( \nabla^2 H \) (corresponding to small changes of the surface on the length scale of the waves), (6.45) describes waves with space dependent velocity

\[
c_p(x,y) = \sqrt{G H(x,y)}
\]

(6.46)

It is obvious that waves slow down if they reach a shallower region. In the mean time their wave length decreases:

\[
\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega} c_p = \frac{2\pi}{\omega} \sqrt{G H} \sim H^{\frac{1}{2}}
\]

(6.47)
If the ground has a small slope, Green’s law can be derived from weakly non-linear theory (see textbooks, e.g. Lamb, *Hydrodynamics*, Cambridge Univ. Press):

\[ A \sim H^{-\frac{1}{4}} \]  

(6.48)

From there one sees that the amplitude of waves increases if they approach the shore. We shall return to this issue in the sect. on Tsunamis.
6.3.4 Generation of waves by a time-dependent ground

The boundary conditions now have the form:

\[ \hat{n} \cdot \vec{v} = \hat{n} \cdot \nabla \Phi = \dot{f} \]

This gives another term (underlined)

\[ \dot{\Phi} = -G(h - 1) - \frac{1}{2} (\nabla_2 \Phi)^2 \quad (6.49) \]
\[ \dot{h} = -(\nabla_2 h) \cdot (\nabla_2 \Phi) - h \Delta_2 \Phi + (\nabla_2 f) \cdot (\nabla_2 \Phi) + f \Delta_2 \Phi + \dot{f} \quad (6.50) \]

If \( \Phi = \text{const} \) (fluid in rest) \( \rightarrow \) \( \dot{h} = \dot{f} \rightarrow h(t) = f(t) + \text{const} \)

- the ground motion is equal to the surface motion
- no time delay, reason: fluid is assumed to be incompressible
- A ground motion may generate waves:
  linearized wave equation with \( H = 1 - f \):

\[ \frac{1}{G} \Phi - \nabla_2 [H \nabla_2 \Phi] = \dot{f} = -\dot{H} \quad (6.51) \]

This is an inhomogeneous wave equation. It can be formally solved using an appropriate Green's function:

\[ \Phi(x,y,t) = \int \int dx' dy' \int dt' D(x-x',y-y',t-t') H(x',y',t') \quad (6.52) \]
Example

cconsider the following localized ground motion:

$$H(x,y,t) = \begin{cases} 
1 + v_0 \cdot t \delta(x) \delta(y), & 0 \leq t \leq t_0 \\
1 + v_0 \cdot t_0 \delta(x) \delta(y), & t_0 < t
\end{cases} \quad (6.53)$$

solution: circular waves

 snapshots at various times $t$
Numerical solution in two dimensions

We chose

\[ f(x,y,t) = a e^{-r^2/\beta^2} \cos \Omega t \]  \hspace{1cm} (6.54)

- oscillating ground localized at \( r = 0 \) with a gaussian distribution.
- slopy ground (ramps in \( x \)-direction, minimum in the center).
6.3.5 Tsunamis

The notion “Tsunami” was coined by Japanese fishermen and means “wave in harbor”. The fishermen went out to the sea during the night for fishing. On their return, they found the harbor destroyed by a flood. Since they didn’t notice anything unusual on the open sea, they thought that these waves were generated in the harbor.

- Tsunamis are caused by seaquakes or landslides
- More than 80 Tsunamis observed in the last 10 years
  - Christmas 2004, Sri Lanka, India, Thailand, more than 200,000 victims
  - Lissabon 1755, caused by the big earthquake 60,000 victims
  - Krakatau 1883, a wave was generated that traveled 7 times round the earth
  - Japan, 1896, a Tsunami called “Sanriku” caused waves with amplitudes up to 23 meters

What is the difference between a Tsunami and waves generated by wind?

Wind accelerates the fluid on a thin layer at the surface of the sea. Waves generated by wind are short waves or deep water waves.

Due to the generation of a Tsunami on the ground of the sea, the whole water column over the seismic center is elevated:

Thus, the fluid over the whole depth moves. Although the fluid motion is rather slow compared to that caused by wind waves, its kinetic energy is enormous due to the large mass in motion.

Waves generated by a seaquake are long waves or shallow water waves.
• On the high seas (off shore) the wave amplitude is very small: 10 - 50 cm
• The wave length is > 100 km
• The water depth is 4 - 7 km
• For Tsunamis, the Shallow-Water theory applies

There we found a relation between phase velocity and water depth:

\[ c = \sqrt{gh_0} \]

If we use \( g = 9.81 \text{ m/s}^2 \) and \( h_0 = 4000 \text{ m} \) we find

\[ c \approx 200 \text{ m/s} \approx 700 \text{ km/h} \]

• A Tsunami may cross an ocean within a few hours!
• There is almost no damping, because the particle velocity is very slow.

\[ v = \frac{A}{h_0} c \]

with \( A = 50 \text{ cm}, h_0 = 4000 \text{ m}, c = 200 \text{ m/s} \) one gets
6.3. THE SHALLOW WATER EQUATIONS

\[ v \approx 2.5 \text{ cm/s} \]

This cannot be measured on the surface, because it is completely covered by the natural motion (wind). Tsunamis can only be detected well below the surface, where the water is usually not moving (or only in large scaled streams).

For the frequency, we can estimate

\[ \nu = \frac{\omega}{2\pi} = \frac{c \cdot k}{2\pi} = \frac{c}{\lambda} \]

Taking \( \lambda = 100 \text{ km} \) and \( c = 200 \text{ m/s} \) one has \( \nu \approx 0.002 \text{ Hz} \), corresponding to \( \Delta t \approx 500 \text{ s} \) between two consecutive waves.

From Green’s law we know that the amplitude increases by approaching the shore:

\[ A \sim H^{-\frac{1}{4}} \]

The water velocity is also a function of the depth:

\[ v = \frac{A}{H} c \]

Since \( c \sim H^{\frac{1}{2}} \) we finally have a rather strong increase of the water velocity while a Tsunami reaches the shores:

\[
\frac{v}{v_0} = \left( \frac{H_0}{H} \right)^{\frac{3}{4}} \tag{6.55}
\]

Taking as an example \( H_0 = 5000 \text{ m} \) (high seas) and \( v_0 = 10 \text{ cm/s} \), this yields at the shore \( H = 10 \text{ m} \) \( v \approx 10 \text{ m/s} \).
Chapter 7

Influence of a slow rotation

The rotation of the earth around its axis through the poles with the angular frequency

$$\Omega = \frac{2\pi}{1 \text{ day}} \approx 7.27 \cdot 10^{-5} \text{ Hz}$$

creates pseudo forces in the rotating frame of references fixed to the surface of the earth. These forces are the centrifugal (or centripetal) force and the Coriolis force. The first one is $\sim \Omega^2$ and can be neglected, the latter one is $\sim \Omega$ and manifests itself e.g. in the motion of Foucault’s pendulum, but also in large scale fluid motion in oceans or in the atmosphere.

In this chapter we shall focus on turbulent motion in thin rotating layers. Both systems, atmosphere and oceans, can be considered as thin layers, since the lateral dimension is much larger than the horizontal one. Turbulence is described by introducing an effective viscosity, the so-called “eddy viscosity”

7.1 Navier-Stokes equations in a rotating system

If the axis of rotation is constant in space, the pseudo forces in the co-rotating system read

$$\vec{F}_p = -2 \frac{\vec{\Omega} \times \vec{v}}{\text{Coriolis}} + \frac{\Omega^2 \vec{R}}{\text{centripetal} \approx 0} \quad (7.1)$$

with

− $\vec{R}$ in the co-rotating frame perpendicular to the axis of rotation
− $\vec{v}$ velocity in the co-rotating frame
Neglecting the centripetal force, the Navier-Stokes eqs. in the rotating frame read

\[
\dot{\vec{v}} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho_0} \nabla p - \frac{g \rho}{\rho_0} \hat{e}_z + \vec{F} - 2 \vec{\Omega} \times \vec{v} \\
\text{div} \vec{v} = 0
\]

The so-called “Boussinesq approximation” is the special case, where all material properties and the density are assumed to be constant. The only exception is the density in the external force term, to account for buoyancy). There one assumes $\rho$ as a linear function of temperature.
7.1. NAVIER-STOKES EQUATIONS IN A ROTATING SYSTEM

7.1.1 Concept of “Eddy Viscosity”

The usual Newtonian friction of a gas (atmosphere) was given as

\[ \vec{F} = \text{div} \sigma \]

with

\[ \sigma_{ij} = \rho \nu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \]

where \( \nu \) is the kinematic viscosity and originates from the exchange of momentum between the molecules of the gas.

Geophysical flows (ocean, atmosphere) are usually turbulent. This leads to the so-called turbulent mixing. Instead of computing the fully turbulent flows (which is still beyond the scope of modern computers) one uses averaged equations (the so-called Reynolds equations). They have the same form than the Navier-Stokes eqs. but a much larger “effective” viscosity, sometimes called eddy viscosity. Turbulence may increase the viscosity by many orders of magnitude.

It is assumed that

- effective viscosity \( \sim \) velocity gradients
- The motion of oceans and atmosphere is not isotrop
- It is organized in shallow stratified layers
- one finds

\[ |\vec{v}_H| \gg |\vec{v}_v| \rightarrow \nu_H \gg \nu_v \gg \nu \]

Estimates for several effective viscosities are:

<table>
<thead>
<tr>
<th>atmosphere (lower)</th>
<th>ocean (upper)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu_v = 10^2 ) m(^2)/s</td>
<td>( \nu_v = 0.01 ) m(^2)/s</td>
</tr>
<tr>
<td>( \nu_H = 10^5 ) m(^2)/s</td>
<td>( \nu_H = 100 ) m(^2)/s</td>
</tr>
<tr>
<td>air: ( \nu = 10^{-5} ) m(^2)/s</td>
<td>water: ( \nu = 10^{-6} ) m(^2)/s</td>
</tr>
</tbody>
</table>

For the stress tensor components, one finds the relations
CHAPTER 7. INFLUENCE OF A SLOW ROTATION

\[ \sigma_{xz} = \sigma_{zx} = \rho v_x \frac{\partial v_x}{\partial z} + \rho v_H \frac{\partial v_z}{\partial x} \]
\[ \sigma_{yz} = \sigma_{zy} = \rho v_y \frac{\partial v_y}{\partial z} + \rho v_H \frac{\partial v_z}{\partial y} \]
\[ \sigma_{xy} = \sigma_{yx} = \rho v_H \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \]
\[ \sigma_{xx} = 2 \rho v_H \frac{\partial v_x}{\partial x} \]
\[ \sigma_{yy} = 2 \rho v_H \frac{\partial v_y}{\partial y} \]
\[ \sigma_{zz} = 2 \rho v_H \frac{\partial v_z}{\partial z} \] (7.2)

and for the friction force components

\[ F_x = v_H \Delta_2 v_x + v \frac{\partial^2}{\partial z^2} v_x, \quad \text{with} \quad \Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]
\[ F_y = v_H \Delta_2 v_y + v \frac{\partial^2}{\partial z^2} v_y \]
\[ F_z = v_H \Delta_2 v_z + v \frac{\partial^2}{\partial z^2} v_z \] (7.3)

- no isotropy \( v_H \neq v_v \)
- motion is mainly horizontal
- concept of thin (coupled) layers for weather models

7.2 Thin layers on a Rotating Sphere

- both atmosphere and oceans can be considered as thin layers

\[ L \gg h_0, \quad L = \text{lateral length scale} \]
\[ h_0 = \text{depth} \]

- if \( L \ll R \), the curvature of the earth can be neglected
- Cartesian system can still be used – but pseudo forces (Coriolis, centrifugal) can be important

Cartesian system on rotating earth:
7.2. THIN LAYERS ON A ROTATING SPHERE

rotation axis in the local system

\[
\Omega = \frac{2\pi}{1 \text{ day}} = \frac{2\pi}{86400 \text{ s}} \approx 7.27 \cdot 10^{-5} \text{ Hz}
\]  

Then the Coriolis force in the local system read

\[
2 \tilde{\Omega} \times \tilde{v} = 2 \Omega \left[ (v_z \cos \vartheta - v_y \sin \vartheta) \hat{e}_x + v_x \sin \vartheta \hat{e}_y + v_y \cos \vartheta \hat{e}_z \right]
\]  

\*\*\*
or in components

\[\begin{align*}
2(\Omega \times \vec{v})_x &= -2\Omega \sin \vartheta \: v_y = -f \: v_y \\
2(\Omega \times \vec{v})_y &= 2\Omega \sin \vartheta \: v_x = f \: v_x \\
2(\Omega \times \vec{v})_z &= -2\Omega \cos \vartheta \: v_x \approx 0 \quad \text{(small compared to gravity)}
\end{align*}\]  

(7.8)

here, \( f \) denotes the \textit{planetary vorticity} or the \textit{Coriolis frequency}

\[ f \equiv 2 \Omega \sin \vartheta \]

It depends on the geographical latitude:

\[ 0 \text{ (equator)} \leq |f| \leq 1.5 \cdot 10^{-4} \text{ Hz (north and south pole)} \]

If we neglect centrifugal forces (good approx. due to \( \sim \Omega^2 \)) we may write down the basic equations for a thin layer in a rotating local system at latitude \( \vartheta \):

\[\begin{align*}
\dot{v}_x + (\vec{v} \cdot \nabla) \vec{v}_x &= f v_y - \frac{1}{\rho_0} \partial_x p + v_H \Delta_2 v_x - v_v \partial_{zz}^2 v_x \\
\dot{v}_y + (\vec{v} \cdot \nabla) \vec{v}_y &= -f v_x - \frac{1}{\rho_0} \partial_y p + v_H \Delta_2 v_y - v_v \partial_{zz}^2 v_z \\
\dot{v}_z + (\vec{v} \cdot \nabla) \vec{v}_z &= -\frac{1}{\rho_0} \partial_z p - g \frac{\rho}{\rho_0} + v_H \Delta_2 v_z - v_v \partial_{zz}^2 v_z
\end{align*}\]

(7.9)  (7.10)  (7.11)

\[ \text{div} \vec{v} = 0 \]  

(7.12)

This has to be supplemented by a material law (state equation)

\[ p = p(\rho) \quad \text{or} \quad p = p(T, \rho) \]

and an equation for the temperature field (from first law of thermodynamics).

The basic equations are written in the so-called \textit{Boussinesq approximation}. This means that all material parameters are assumed to be constant except for the density in the gravity force term in (7.11). There one usually assumes a linear dependence \( \rho \sim T \) to account for buoyancy.

To summarize, we used the following approximations for the basic equations:

- concept of eddy viscosities
7.3. GEOSTROPHIC FLOWS

- Motion organized in thin layers
- neglection of curvature
- neglection of centrifugal force
- Boussinesq approximation

7.3 Geostrophic Flows

- Now consider a stationary and inviscid (perfect) flow \((\nu_H = \nu_v = 0)\)
- pressure gradient has to be balanced by Coriolis force

Estimate of acceleration forces:

\[
\left| \frac{(\vec{v} \cdot \nabla)\vec{v}}{\bar{\Omega} \times \vec{v}} \right| = \frac{\text{Nonlinear acceleration}}{\text{Coriolis force}} = R_0 \quad (7.13)
\]

\(R_0\): Rossby number \(\sim \frac{U^2/L}{f \cdot U}\)
- \(U\): typical horizontal velocity
- \(L\): horizontal length scale

example (lower) atmosphere:

\[
U \approx 10 \text{ m/s}, \quad f \approx 10^{-4} \text{ Hz}, \quad L = 10^6 \text{ m} \quad \Rightarrow \quad R_0 \approx 0.1
\]

In oceans, the Rossby number is even smaller.
- non-linearities can be neglected!

\[
\begin{align*}
fv_y &= \frac{1}{\rho_0} \frac{\partial p}{\partial x} \\
fv_x &= -\frac{1}{\rho_0} \frac{\partial p}{\partial y}
\end{align*}
\]

\(fv\): Geostrophic velocity, geostrophic wind

The geostrophic equations break down, if
• $f \to 0$, i.e. close to the equator
• friction cannot be neglected
• flows are faster or/and on a smaller horizontal scale (e.g. hurricanes)

Geostrophic flows have important properties:

• in a geostrophic flow, velocity is always parallel to the isobars. In a non-rotating system, the velocity is parallel to the pressure gradient, this is perpendicular to the isobars.
• $P(x,y)$ can be regarded as stream function. Prove:

$$\vec{v} \cdot \nabla p = \rho_0 f (v_x v_y - v_y v_x) = 0 \implies \text{isobars = streamlines}$$

- On the northern hemisphere the flow around depressions circulate counterclockwise
- flow around highs circulate clockwise
• On the southern hemisphere its vice versa

– From the isobars of a weather chart, one can directly conclude on the winds.

### 7.4 Thermal wind

Up to now, we examined the case of an isothermal atmosphere. But usually, besides a vertical gradient, one finds large horizontal gradients from the temperature differences between the equator and the poles.
we assume some horizontal \((y)\) temperature gradient according to

\[
T(y) = T_0 + y' (T_1 - T_0)
\]

With \(T_0 > T_1\). Further, we approximate the atmosphere as ideal gas.

\[
\rho(p, T) = \frac{p \cdot m}{R \cdot T}
\]

with \(R = \) gas constant and \(m = \) molar mass.

The pressure follows from the hydrostatic balance \((v_z \approx 0)\)

\[
\frac{\partial p}{\partial z} = -g \cdot \rho = -g \cdot \frac{m \cdot p}{R \cdot T(y)}
\]

This equation can be integrated and gives:

\[
p(y, z) = p_0 e^{-\alpha z / T(y)}, \quad \text{with} \quad \alpha = \frac{g \cdot m}{R} \quad \text{and} \quad p_0 = p(z = 0)
\]

The isobars are found from \(p = \) const and read

\[
z = c_0 \cdot T(y)
\]

The density distribution follows from the state equation and takes the form

\[
\rho = \frac{p \cdot m}{R \cdot T(y)} = \frac{p_0 \cdot m}{R \cdot T(y)} e^{-\alpha z / T(y)}
\]

The isochores \((\rho = \) const\) read
The thermal wind (velocity) follows from the geostrophic eqs:

\[ v_x = -\frac{1}{\rho_0 f} \frac{\alpha \cdot z}{T^2} \left( \frac{\partial T}{\partial y} \right) \cdot p(y, z) > 0 \]  \hspace{1cm} (7.16)

\[ v_y = \frac{1}{\rho_0 f} \frac{\partial_y p}{\partial x} = 0 \]  \hspace{1cm} (7.17)

- wind blows outwards of the yz-plane
- for small \( z \) it increases with height
- this is the west drift
W summarize the thermal wind equations

\begin{align}
\begin{aligned}
v_y &= \frac{1}{\rho_0 f} \frac{\partial x}{\partial z} p \\
v_x &= -\frac{1}{\rho_0 f} \frac{\partial y}{\partial z} p
\end{aligned}
\end{align}
\]  \text{geostrophic flow} \quad (7.18)

\[0 = -\partial_z p - g \rho \quad \text{hydrostatic balance}\]

\[\rho = \rho(p, T) \quad \text{state equation}\]

We derivate the first two eqs. by \(z\) and use the third one:

\begin{align}
\frac{\partial v_y}{\partial z} &= \frac{1}{\rho_0 f} \frac{\partial x}{\partial z} \frac{\partial x}{\partial p} = -\frac{g}{\rho_0 f} \frac{\partial x}{\partial z} \frac{\partial \rho}{\partial z} \\
\frac{\partial v_x}{\partial z} &= -\frac{1}{\rho_0 f} \frac{\partial y}{\partial z} \frac{\partial y}{\partial p} = \frac{g}{\rho_0 f} \frac{\partial y}{\partial z} \frac{\partial \rho}{\partial z}
\end{align}
\]  \quad (7.19)

- a horizontal temperature gradient can cause a vertical velocity gradient via a density gradient

- the vector \(\partial_z \vec{v}_H\) is directed perpendicular to \(\text{grad} \rho\)

### 7.5 The Shallow Water equations with rotation

We ask for the influence of earth rotation on gravity waves in oceans.
We start with the kinematic boundary condition

$$
\dot{h} = v_z \bigg|_{h_0 + \eta} - v_x \partial_x \eta - v_y \partial_y \eta
$$

(7.20)

In the shallow water theory, the horizontal velocity components do not depend on \(z\) (in lowest order). By integrating the continuity equation

$$
\int_0^{h_0 + \eta} d\zeta \quad \text{div} \bar{\nu} = 0
$$

(7.21)

one finds \(v_z\) at \(z = h_0 + \eta\):

$$
\left. v_z \right|_{h_0 + \eta} = - (\partial_x v_x + \partial_y v_y) \cdot (h_0 + \eta)
$$

(7.22)

Inserting this into (7.20) yields

$$
\dot{h} = - \partial_x (v_x \cdot (h_0 + \eta)) - \partial_y (v_y \cdot (h_0 + \eta))
$$

(7.23)

and the Euler eq.

$$
\begin{align*}
\dot{v}_x + v_x \partial_x v_x + v_y \partial_y v_x &= -g \partial_x \eta + f v_y \\
\dot{v}_y + v_x \partial_x v_y + v_y \partial_y v_y &= -g \partial_y \eta - f v_x
\end{align*}
$$

(7.24) (7.25)

where we used

$$
\partial_x p = \rho g \partial_x \eta, \quad \partial_y p = \rho g \partial_y \eta
$$
Eqs. (7.23,7.24,7.25) are the Shallow Water equations in a rotating fluid layer. Note that we didn’t use \( \vec{v} = \nabla \Phi \) as in chapter 6. The reason is that the Coriolis force does not allow for a potential flow.

### 7.5.1 Dispersion relation

The linearized Shallow Water equations read:

\[
\begin{align*}
\dot{\eta} &= -h_0 (\partial_x v_x + \partial_y v_y) \\
\dot{v}_x &= f v_y - g \partial_x \eta \\
\dot{v}_y &= -f v_x - g \partial_y \eta
\end{align*}
\] (7.26)

To find the dispersion relation of gravity waves with rotation, we make the plane wave ansatz

\[
\vec{k} = \begin{pmatrix} k \\ \ell \end{pmatrix}
\]

Using

\[
\begin{pmatrix} \eta \\ v_x \\ v_y \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} e^{i(kx + \ell y - \omega t)}
\] (7.27)

in (7.26) one gets the linear algebraic system

\[
\begin{pmatrix}
-\omega, & ikh_0, & ilh_0 \\
ike, & -\omega, & -f \\
ilg, & f, & -i\omega
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix} = 0
\] (7.28)

From the solvability condition: \( \det(\ldots) = 0 \), one finds

\[
\omega^2 = f^2 + h_0 g |k|^2
\] Dispersion relation
It is obvious that rotation can be neglected for waves with
\[ \sqrt{h_0 g} |k| \gg |f| \]
or
\[ \lambda \ll 4\pi \sqrt{h_0 g} \frac{1}{|f|} \]
using the values for the deep ocean in middle latitudes \( h_0 \approx 5 \text{ km} \) and \( f = 10^{-4} \text{ Hz} \), one finds that only very long waves are affected by the Coriolis force with \( \lambda > 3000 \text{ km} \).

\[ c = \frac{\omega}{k} \]

\[ \eta = A \cos(\omega t - kx) \quad (7.29) \]

One finds the velocity from the linearized Shallow Water eqs:
\[
\begin{align*}
    v_x &= \frac{\omega A}{h_0} \cos(kx - \omega t) \\
    v_y &= -\frac{f A}{kh_0} \sin(kx - \omega t)
\end{align*}
\]

ellipses

\[ (7.30) \]
7.5.3 Large scale motion (inertial motion)

consider very long waves

\[ k \to 0 \quad \omega \to f \quad \eta \to 0 \]

Shallow Water eqs.:

\[
\begin{align*}
\dot{v}_x &= f v_y \\
\dot{v}_y &= -f v_x
\end{align*}
\]

have the solution

\[
v_x = v_0 \cos(ft), \quad v_y = v_0 \sin(ft) \quad (7.31)
\]

\[ \Rightarrow \] particles move on big circles with

\[ T = \frac{2\pi}{f} \]

as “inertial period”, which depends on \( \vartheta \).

The radius \( R \) of these circles can be found from the balance of Coriolis and centrifugal forces (or simply by integration of (7.31)).

\[
\frac{R \cdot \omega^2}{\text{centrif.}} = R \cdot f^2 = \frac{f v_0}{\text{Coriolis}} \quad \Rightarrow \quad R = \frac{v_0}{f}
\]
As an example, let

\[ v_0 \approx 0.1 \text{ m/s} \quad \rightarrow \quad R \approx 1 \text{ km} \]

### 7.5.4 Waves along a coastline (Kelvin waves)

under a crest, particles move with \( v_x > 0 \) (in direction of \( \vec{k} \))

under a trough, particles move with \( v_x < 0 \)

If the velocity goes south-north (or vice versa), the Coriolis force accelerates the water perpendicular to the direction of propagation. If there is no transversal flow, the Coriolis force must be balanced by a pressure gradient in \( y \) direction. This can be reached by an inclined surface, surface gradient in \( y \).
\[ \dot{v}_y = 0 \quad \rightarrow \quad f v_x = -g \partial_y \eta \quad \rightarrow \quad \partial_y \eta < 0 \quad (\text{crest}) \]
\[ \partial_y \eta > 0 \quad (\text{trough}) \]

To compute the surface profile near the coast, we write down the Shallow Water equations with \( v_y = 0 \):

\[ \dot{\eta} = -h_0 \partial_x v_x \]
\[ \dot{v}_x = -g \partial_x \eta \]
\[ f v_x = -g \partial_y \eta \]

The first two eqs. are similar to the Shallow Water eqs. without rotation. As solution one finds waves in \( x \)-direction (along the coast) with

\[ \omega = ck, c = \sqrt{gh_0} \]

Such a wave is named “Kelvin wave”. But if it moves along the coast, the transverse velocity \( (v_y) \) must vanish and the last eq. can only be solved by a \( y \)-dependent surface:

\[ \dot{\eta} = -h_0 \partial_x v_x = \frac{h_0 g}{f} \partial_x \partial_y \eta \]

we use the ansatz

\[ \eta = \xi(y) e^{-i \omega t} e^{ikx} \]

and find

\[ \partial_y \xi + \frac{f}{c} \xi = 0 \]

which is solved by

\[ \xi(y) = \xi_0 e^{-\frac{f}{c} y} \]

\[ \eta(y) \]

coast
From that we find \( \lambda = \frac{c}{f} \) as transverse decay of the Kelvin wave. In oceans this can be \( \lambda \approx 10^6 \) m.

As a consequence of the calculations above, we see that the coast must be on the right hand side of the Kelvin wave (on the northern hemisphere).

### 7.5.5 Gulf stream

The influence of the Coriolis force can be seen on the location of the Gulf stream. On the southern hemisphere it is shifted to the west, on the northern to the east. In the equatorial zone it crosses the atlantic ocean.
Chapter 8

Solitons

8.1 Discovery

- discovered by John Scott Russell in 1834

- localized states of elevated (or depressed) surface

- only in one spatial dimension possible (narrow channels)

- certain specific properties concerning speed, length, interaction

The discovery of John Scott Russell (in his own words)

“I believe I shall best introduce the phenomenon by describing the circumstances of my own first acquaintance with it. I was the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot an a half in height. It’s height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation, a name which it now very generally bears.”
Combination of linear and nonlinear behavior

\[ A \sim v \]
\[ \ell \sim \frac{1}{\sqrt{v}} \sim \frac{1}{\sqrt{A}} \]

The shapes (a) are conserved, even after (c) a collision (b) of two solitons.

Experiments of Russell:

\[ w = \sqrt{g(h_0 + a)} \quad (8.1) \]

\( a \to 0 \): relation for shallow water waves.
but: non-linearities are important (small corrections).

60 years later: Solitons are solutions of the Korteweg-de Vries-equation.

### 8.2 The Korteweg – de Vries equation

derived by D.J. Korteweg and G. de Vries (1895) from the Euler equations in a systematic way.

The starting point are the shallow water equations:

1st step

we use

\[ h = 1 + \alpha \eta, \quad \Phi = \alpha \bar{\Phi} \]

\( \alpha \) is a small parameter (amplitude), \( \alpha \ll 1 \)

\[
\begin{align*}
\dot{\bar{\Phi}} + G \eta &+ \frac{\alpha}{2} (\partial_x \bar{\Phi})^2 + \frac{\alpha}{2\delta^2} (\partial_x \bar{\Phi})^2 = 0 \\
\partial_z \bar{\Phi} - \delta \dot{\eta} - \delta^2 \alpha (\partial_x \bar{\Phi})(\partial_x \eta) & = 0 \\
\partial_{zz} \bar{\Phi} + \delta^2 \partial_{xx} \bar{\Phi} & = 0
\end{align*}
\]  

(8.2)

(in the following we omit the bars).

for \( \alpha = 0 \), one finds the linear wave equation

\[
\partial_{xx} \Phi - \frac{1}{G} \Phi = 0 \quad \rightarrow \quad \text{waves with} \quad c_0 = \pm \sqrt{G}
\]

(8.3)

2nd step

assume \( \alpha = \delta^2 \)

3rd step

change to moving frame \( x = \tilde{x} - c_0 t \)
\[ \Phi = \Phi(x - c_0 t, \alpha t), \quad \eta = \eta(x - c_0 t, \alpha t) \quad (8.4) \]
\[ \Phi = \alpha \partial_t \Phi - c_0 \partial_x \Phi, \quad \dot{\eta} = \alpha \partial_t \eta - c_0 \partial_x \eta \quad (8.5) \]

Inserting this into eqs. (8.2)

\[ \alpha \dot{\Phi} - c_0 \partial_x \Phi + G \eta + \frac{\alpha}{2} (\partial_x \Phi)^2 + \frac{1}{2} (\partial_z \Phi)^2 = 0 \quad (8.6) \]
\[ \partial_z \Phi - \alpha^2 \dot{\eta} + \alpha c_0 \partial_x \eta - \alpha^2 (\partial_x \Phi)(\partial_x \eta) = 0 \quad (8.7) \]
\[ \partial_{zz} \Phi + \alpha \partial_{xx} \Phi = 0 \quad (8.8) \]

4th step

Expansion with respect to \( \alpha \)

\[ \eta = \eta^{(0)} + \alpha \eta^{(1)} + \alpha^2 \eta^{(2)} + ... \quad (8.9) \]
\[ \Phi(x, z, t) = \Phi^{(0)}(x, t) + \alpha \left[ -\Phi^{(0)\prime\prime} \cdot \frac{z^2}{2} + f_4(x, t) \right] \]
\[ + \alpha^2 \left[ \Phi^{(0)\prime\prime\prime} \cdot \frac{z^4}{24} - f_4'' \cdot \frac{z^2}{2} + f_6(x, t) \right] + ... \quad (8.10) \]

(for the latter expansion see sect. 6.3, eq. (6.36)).

5th step

Compare orders of \( \alpha^n \)

\[ \alpha^0 \]

\[ eq.(8.6) \rightarrow G \eta^{(0)} - c_0 \partial_x \Phi^{(0)} = 0 \]

We use \( G = c_0^2 \):

\[ \partial_x \Phi^{(0)} = c_0 \eta^{(0)} \]

\[ \alpha^1 \]

\[ eq.(8.6) \rightarrow c_0 \eta^{(1)} - \partial_x f_4 + \frac{1}{c_0} \Phi^{(0)} + \frac{1}{2} \partial_{xxx} \Phi^{(0)} + \frac{1}{2c_0} (\partial_x \Phi^{(0)})^2 = 0 \]
8.3. Numerical solutions of the KdV equation

In this sect. we shall introduce the standard numerical method for the KdV.

This is the Korteweg-de Vries equation (KdV).

In addition, one needs boundary conditions and initial conditions for \( \eta \).

Solution of KdV: solitons (and more), see next sects.
8.3.1 Conserved quantities

It is important to know that there are certain quantities, which are conserved.

Consider

$$\bar{\eta}(t) = \frac{1}{L} \int_{0}^{L} \eta(x, t) \, dx$$  \hspace{1cm} (8.11)

the mean height. If the total mass (of an incompressible fluid) is conserved, this must be true also for its volume. Since the volume is proportional to the mean height, we have

$$\bar{\eta} = \text{constant}$$

Of course this is also a feature of the KdV:

$$\dot{\bar{\eta}} = \frac{\partial}{\partial t} \left( \frac{1}{L} \int_{0}^{L} \eta(x, t) \, dx \right) = \frac{1}{L} \int_{0}^{L} \dot{\eta}(x, t) \, dx$$

$$\overset{(\text{KdV})}{=} - \frac{1}{L} \int_{0}^{L} \partial_{xxx}^{3} \eta \, dx - \frac{1}{L} \int_{0}^{L} \eta \partial_{x} \eta \, dx$$

$$= - \left. \frac{1}{L} \partial_{xx}^{2} \eta \right|_{0}^{L} - \frac{1}{2} \left. \eta^{2} \right|_{0}^{L} = 0$$  \hspace{1cm} (8.12)

where periodic lateral boundary conditions have been assumed.

Another conserved quantity is

$$\bar{E}(t) \sim \int_{0}^{L} \eta^{2}(x, t) \, dx$$

remember that

$$\dot{\partial_{x} \Phi} = v_{x} = c_{0} \bar{\eta}$$

and so

$$\eta^{2} = \frac{1}{c_{0}^{2}} v_{x}^{2} \sim E + 0(\delta^{2})$$

Shallow water approx

where $E$ is the kinetic energy density. Then, $\bar{E}$ corresponds to the total kinetic energy and must be also constant in time:
8.3. NUMERICAL SOLUTIONS OF THE KDV EQUATION

\[ \dot{E} \sim 2 \int_0^L \eta \dot{\eta} \, dx = -2 \int_0^L \eta \partial_x^3 \eta \, dx - 2 \int_0^L \eta^2 \partial_x \eta \, dx \]
\[ = -2 \int_0^L \partial_x[(\eta \partial_x^2 \eta) - \frac{1}{2} (\partial_x \eta)^2] \, dx - \frac{2}{3} \int_0^L \partial_x(\eta^3) \, dx \]
\[ = 0 \quad (8.13) \]

- There are even more conserved quantities
- The numerical method used for integration of the KdV eq. should keep these quantities constant in time

8.3.2 Spectral methods

One approximates the solution \( \eta \) by a finite sum of linearly independent functions \( \varphi_{\ell} \):

\[ \tilde{\eta}(x,t) = \sum_{\ell=1}^N \xi_{\ell}(t) \varphi_{\ell}(x) \quad (8.14) \]

The \( \varphi_{\ell}(x) \) are called basic functions or basis. These could be polynomials, trigonometric functions, etc. If the basis is complete, one has

\[ \tilde{\eta} \to \eta \quad \text{if} \quad N \to \infty \]

The equation which we wish to solve numerically (e.g. the KdV eq.) may have the general form

\[ \dot{\eta} = F(\eta) \quad (8.15) \]

Inserting (8.14), one finds

\[ R(x,t) = \dot{\tilde{\eta}} - F(\tilde{\eta}) \quad (8.16) \]

where \( R \) is called residual. One can determine the amplitudes \( \xi_{\ell} \) requiring that the residual is always perpendicular to all basic functions. This method ensures that
and is called \textit{Galerkin method}. Thus

\[
\int R \varphi_\ell(x) \, dx \xrightarrow{\ell \to 0} \int \varphi_\ell \dot{\eta} \, dx - \int \varphi_\ell F(\eta) \, dx = \sum_{\ell'} \int \xi_{\ell'} \varphi_\ell \varphi_{\ell'} \, dx - \int \varphi_\ell F(\eta) \, dx
\]

or

\[
\sum_{\ell'} a_{\ell \ell'} \dot{\xi}_{\ell'} = F_\ell(\xi_1, \ldots, \xi_N) \quad (8.17)
\]

with

\[
a_{\ell \ell'} = \int \varphi_\ell \varphi_{\ell'} \, dx, \quad F_\ell = \int \varphi_\ell F \, dx
\]

If the \( \varphi_\ell \) are orthonormal, then \( a_{\ell \ell'} = \delta_{\ell \ell'} \) and

\[
\dot{\xi}_\ell = F_\ell(\xi_1, \ldots, \xi_N) \quad \ell = 1 \ldots N \quad (8.18)
\]

With (8.18) one has to solve a system of \( N \) ordinary differential equations instead of one partial differential equation (8.15).

### 8.3.3 Finite difference methods

The finite difference methods (FD) compute the solution function only at certain (not always equally spaced) points in real space:

\[
\eta(x)\text{ with } x_n = n \cdot \Delta x
\]

one can write

\[
\eta(x, t) = \eta(x_n, t) = \eta_n(t)
\]
8.3. NUMERICAL SOLUTIONS OF THE KDV EQUATION

The spatial derivatives are approximated by finite difference formulas. Here, several orders in $\Delta x$ can be used, leading to methods with different accuracy, but also with different effort and efficiency. For instance

\[
\begin{align*}
\partial_x \eta(x_n) &\approx \frac{\eta_{n+1} - \eta_{n-1}}{2\Delta x} \\
\partial^2_{xx} \eta(x_n) &\approx \frac{\eta_{n+1} - 2\eta_n + \eta_{n-1}}{\Delta x^2} \\
\partial^3_{xxx} \eta(x_n) &\approx \frac{\eta_{n+2} - 2\eta_{n+1} + 2\eta_{n-1} - \eta_{n-2}}{2\Delta x^3}
\end{align*}
\]

(8.19) (8.20)

e tc. \ldots

Finally one gets from (8.15) (for the spatial case of the KdV eq.)

\[
\dot{\eta}_\ell = F(\eta_{\ell-2}, \eta_{\ell-1}, \eta_{\ell}, \ldots, \eta_{\ell+2}) , \quad \ell = 1 \ldots N
\]

again a system of $N$ ordinary differential equations.

8.3.4 Time integration

The remaining task is to solve a large system of ordinary diff. eqs. of the general form

\[
\dot{\eta}_\ell = F(\eta_1 \ldots \eta_N) \quad \ell = 0 \ldots N
\]

(8.21)

This is done by discretizing the time

\[t_n = n \cdot \Delta t\]

with the time step $\Delta t$. We use the notation

\[
\eta_\ell(t) = \eta_\ell(t_n) \equiv \eta^n_\ell
\]

To approximate the time derivative in (8.21), several possibilities are in order. We concentrate on one-step methods. They have the lowest accuracy but the highest efficiency.
A. Euler forward (explicit)

Taking
\[ \dot{\eta}_n^\ell \approx \frac{(\eta_{n}^{\ell+1} - \eta_{n}^{\ell})}{\Delta t} \]  
(8.22)

leads to the Euler forward scheme

\[ \eta_{n}^{\ell+1} = \eta_{n}^{\ell} + \Delta t \cdot F_{\ell}(t_n) \]

B. Euler backward (implicit)

with
\[ \dot{\eta}_n^\ell \approx \frac{(\eta_{n}^{\ell} - \eta_{n-1}^{\ell})}{\Delta t} \]

one finds the Euler-backward scheme

\[ \eta_{n}^{\ell} = \eta_{n-1}^{\ell} + \Delta t \cdot F_{\ell}(t_n) \]  
(8.23)

- drawback: (8.23) has to be solved for \( \eta_{n}^{\ell} \)
- advantage: numerical stability can be much better

C. Crank-Nicolson

A combination of forward and backward methods leads to the so-called Crank-Nicolson scheme. It is of higher order in \( \Delta t \) (better accuracy) and reads

\[ \eta_{n}^{\ell+1} = \eta_{n}^{\ell} + \frac{1}{2} \Delta t \left( F_{\ell}(t_n) + F_{\ell}(t_{n+1}) \right) \]

D. Leap-frog

Taking instead of (8.22) the formula
\[ \dot{\eta}_n^\ell \approx \frac{\eta_{n}^{\ell+1} - \eta_{n-1}^{\ell-1}}{2\Delta t} \]

one arrives at the scheme
\[ \eta^{n+1}_i = \eta^{n-1}_i + 2\Delta t F_i(t_n) \]

This is called leap-frog method.

### 8.3.5 Method of Zabusky and Kruskal

The solution of the periodic boundary-value problem for the KdV equation (after Zabusky & Kruskal, 1965). Initial profile at \( t = 0 \) (dotted line); profile at \( t = 1/\pi \) (broken line); profile at \( t = 3.6/\pi \) (full line).

Numerically found time evolution as solution of the KdV eq., from P. G. Drazin & R. S. Johnson: Solitons: an introduction

This is a special method to solve the kdv equation:

\[ \dot{\eta} = -\partial_{xxx}^3 \eta - \eta \partial_x \eta \]

developed in 1965. Use of the Leap-frog method yields:
\[ \eta_{\ell}^{n+1} = \eta_{\ell}^{n-1} - \frac{1}{3} \left( \eta_{\ell+1}^{n} + \eta_{\ell}^{n} + \eta_{\ell-1}^{n} \right) \cdot \frac{\left( \eta_{\ell+1}^{n} - \eta_{\ell-1}^{n} \right)}{\Delta x} \cdot \Delta t \]

\[ - \left( \eta_{\ell+2}^{n} + 2\eta_{\ell+1}^{n} - 2\eta_{\ell-1}^{n} - \eta_{\ell-2}^{n} \right) \cdot \Delta t \]

(8.24)

both, “mass” \[ \sum_\ell \eta_{\ell}^{n} \] and “energy” \[ \sum_\ell (\eta_{\ell}^{n})^2 \] are conserved to \( O(\Delta t^2) \) !

### 8.4 Analytical one-soliton solution

In this section we show the classical one-soliton solution which can be found analytically.

We search for a solution which moves with a constant velocity \( v \), say to the right.

If the shape is constant, then

\[ \eta(x,t) = u(x - vt) = u(\tilde{x}) \]

\( \tilde{x} \) is the coordinate in the co-moving frame with \( v \) (Galilei transform)

\[ x = \tilde{x} + vt, \quad \tilde{x} = x - vt, \quad t = \tilde{t} \]

For the kdV equation, we express the derivatives in the co-moving frame
8.4. ANALYTICAL ONE-SOLITON SOLUTION

\[ \partial_x = \partial_{\tilde{x}}, \quad \partial_t = \left( \frac{\partial \tilde{x}}{\partial t} \right)_l \partial_{\tilde{t}} + \left( \frac{\partial \tilde{x}}{\partial t} \right)_{-v} \partial_{\tilde{t}} = \partial_{\tilde{t}} - v \partial_{\tilde{t}} \]

Using this, the KdV eq. for \( u \) reads (we omit all wiggles):

\[
\left( \partial_{\tilde{t}} - v \partial_{\tilde{x}} \right) u = -\partial^3_{xxx} u - u \partial_x u = 0.
\]

Since we are only interested in stationary solutions in the co-moving frame, we put the time derivative to zero. Thus one has (prime denotes derivative by \( x \)):

\[
v u' - u''' - uu' = 0 \quad (8.25)
\]

This is an ordinary DEQ in for \( u(x) \). A first integral is easy to find and reads:

\[
v u - u'' - \frac{1}{2} u^2 = c \quad (8.26)
\]

Where the integration constant \( c \) is fixed using boundary conditions \( \tilde{x} \rightarrow \pm \infty \). One may require (for localized pulses)

\[
u, u', u'' \rightarrow 0 \quad \text{for} \quad x \rightarrow \pm \infty
\]

this gives \( c = 0 \) and

\[
v u - u'' - \frac{1}{2} u^2 = 0 \quad (8.27)
\]

or

\[
u u'' = v u - \frac{1}{2} u^2 \quad (8.28)
\]

This is analogue to a one-dimensional motion of a particle in a potential! We identify

\[
x \leftrightarrow t, \quad u \leftrightarrow X
\]

and have

\[
\dot{X} = F(X) = -\frac{dU}{dX}
\]

where \( U(X) \) serves as a potential function (potential energy). One finds
\[ U(X) = -\int F(X) \, dX = -\int (\nu X - \frac{1}{2} X^2) \, dX = -\frac{1}{2} \nu X^2 + \frac{1}{6} X^3 \]

Potential landscape

motion

ampitude ~ velocity, smaller solitons are slower
Phase space

fixed point (saddle)

FP

X

diverging (unphysical) solution

homoclinic orbit (soliton)

waves

\[ \dot{X} \]

\[ X, u \]
Part III

Instabilities
Chapter 9

Concepts

9.1 Exchange of stability

At an instability point (critical point, threshold, bifurcation point), an old state gets unstable, a new one (or more) becomes stable. This new state is qualitatively different (structure, time behavior) from the old one.

Examples are: Ferro magnet, super conductivity, thermodynamic phase transitions gas/liquid or liquid/solid.

Bifurcation, 2nd order phase transition

In many phase transitions, hysteresis or bistability can be found. Then two qualitatively different
states are stable for the same control parameters. A new state may occur, while the old one is still stable:

backward bifurcation, 1st order phase transition

9.2 A model from Classical Mechanics

We consider a ball in a potential landscape. Exchange of stability takes place if the potential changes its structure:

• $\varepsilon$ is the control parameter (bifurcation parameter)
9.3. LINEAR STABILITY ANALYSIS

- stationary states (stable and unstable) are found from

\[ \frac{dV}{dx} = 0 \]  

(9.1)

- The stationary states are called (stable or unstable) fixed points

We compute the old and new states by equation (9.1)

\[ \frac{dV}{dx} = -2\varepsilon x + 4x^3 = 0 \ \Rightarrow \ x_1 = 0, \ x_{2,3} = \pm \sqrt{\varepsilon} \]  

(9.2)

### 9.3 Linear stability analysis

Next question: How can the stability of \( x_i \) be computed?

Idea: perturb the system and see if the old state is reached again (stable) or not (unstable)

- Perturbation decreases: stable fixed point
- Perturbation increases: unstable fixed point
9.3.1 One equation

We need an equation of motion to find \( x(t) \). For our example of the ball in a potential landscape this comes from classical mechanics (overdamped motion):

\[
\frac{m \ddot{x}}{\alpha} + \alpha \dot{x} = F = -\frac{dV}{dx}
\]  
\[\approx 0 \quad (\alpha \text{ large}).\]  

\[\alpha \dot{x} = -\frac{dV}{dx} = 2\varepsilon x - 4x^3\]  

\[x_{1,2,3} = 0, \pm \sqrt{\frac{\varepsilon}{2}} \text{ are stationary states (fixed points) with } \dot{x} = 0\]

now add a small perturbation

\[x(t) = x_i + u(t)\]

and insert this into (9.4). One has in linear order of \( u \)

\[\alpha \dot{u} = 2 \varepsilon (x_i + u) - 4 (x_i + u)^3 = 2 \varepsilon x_i + 2 \varepsilon u - 4 x_i^3 - 12 x_i^2 u - 12 x_i u^2 - 4 u^3\]  
\[\approx 0 \quad \approx 0\]

(the first and the third terms on the right-hand-side cancel, why?) and finally

\[\alpha \dot{u} = (2 \varepsilon - 12 x_i^2)u\]  

- linear ordinary differential equation
- can be solved by

\[u(t) = u_0 e^{\lambda t}\]  

- perturbation behaves exponentially in time
- \( \lambda > 0, u \) increases, the old solution \( x_i \) is unstable
- \( \lambda < 0, u \) decreases, the old solution \( x_i \) is stable

Inserting (9.6) into (9.5) one finds \( \lambda \)
9.3. LINEAR STABILITY ANALYSIS

\[ \lambda = \frac{1}{\alpha} \left( 2 \varepsilon - 12 x_i^2 \right) \]  

(9.7)

Now we evaluate \( \lambda \) for the three fixed points \( x_{1,2,3} \) and find

\[ x_1 = 0 \quad \rightarrow \quad \lambda = 2 \alpha \varepsilon, \quad \text{unstable if } \varepsilon > 0 \]
\[ \text{stable if } \varepsilon < 0 \]

\[ x_{2,3} = \sqrt{\frac{\varepsilon}{2}} \quad \rightarrow \quad \lambda = -\frac{4 \varepsilon}{\alpha}, \quad \text{stable if } \varepsilon > 0 \]
\[ \text{not existing if } \varepsilon < 0 \]

Thus one has an exchange of stability if \( \varepsilon \) passes zero.

---

9.3.2 General method, many equations

Consider the system

\[ q_\ell(t) = N_\ell(q_1 \ldots q_n), \quad \ell = 1 \ldots n \]  

(9.8)

1st step
Determine the fixed point(s)
2nd step
Linearisation around the fixed points
Inserting
\[ q_\ell(t) = q_\ell^0 + u_\ell(t) \]
into (9.8) yields
\[ \dot{u}_\ell = N_\ell(q_1^0 + u_1, \ldots, q_n^0 + u_n) \]
Taylor expansion of the right hand side \((u_\ell\) is small) gives
\[ \dot{u}_\ell = N_\ell(q_1^0, \ldots, q_n^0) + \sum_k \frac{\partial N_\ell}{\partial q_k} \bigg|_{q_0} u_k + \ldots \]
or for short
\[ \dot{u}_\ell = \sum_k L_{\ell k} u_k \quad (9.9) \]
where
\[ L_{\ell k} = \frac{\partial N_\ell}{\partial q_k} \bigg|_{q_0} \]
is the Jacobi Matrix.

3rd step
Solution of the linear eigenvalue problem
Inserting
\[ u_\ell(t) = \varphi_\ell e^{\lambda t} \]
into (9.9) yields the linear eigenvalue problem
\[ \lambda \varphi_\ell = \sum_k L_{\ell k} \varphi_k \]
or
\[ (L - \lambda_j I) \Bar{\varphi}^j = 0 \quad j = 1 \ldots n \quad (9.10) \]
- \(\lambda_j\) are the eigenvalues of the Jacobi Matrix
- \(\Bar{\varphi}^j\) are the eigenvectors of the Jacobi Matrix
9.4 PATTERN FORMATION

The system (9.10) has only then non-trivial solutions, if its system determinant vanishes. From that condition

\[ \det(L - \lambda_j J) = 0 \]

one determines the \( \lambda_j \) which are in general complex valued.

4th step

Stability and unstable manifold

- if all Re(\( \lambda_k \)) < 0 then \( \vec{q}^0 \) is stable
- if only one (or more) Re(\( \lambda_k \)) > 0 then \( \vec{q}^0 \) is unstable
- the “direction” in which the fixed point \( \vec{q}^0 \) is left in \( \vec{q} \)-space is given by the eigenvector \( \vec{\phi}^j \) that belongs to Re(\( \lambda_{ij} \)) > 0. If more than one eigenvalue has a positive real part, the eigenvectors spawn a subspace in \( \vec{q} \)-space, called unstable manifold.

9.4 Pattern Formation

Many patterns found in nature (and physics) are self-organized. They emerge as instabilities from qualitatively different, usually more simple states.

Examples:

- Chemistry: Belousov-Zhabotinsky reaction, CIMA reaction
- Fluids: Convection cells, nonlinear waves, highs and lows, hurricanes
- Biology: cells, nerves, neural nets, skin and hair coating

Alan Turing put (and answered) the following question (1952):

How can spatially periodic structures emerge by self-organization?

A. Turing: Take two species (reactants) that interact:

\[
\begin{align*}
\dot{n}_1 &= a_1 n_1 + a_2 n_2 + D_1 \Delta n_1 + H.O.T. \\
\dot{n}_2 &= b_1 n_1 + b_2 n_2 + D_2 \Delta n_2 + H.O.T.
\end{align*}
\] (9.11)
where \( H.O.T. \) denotes “higher order terms” (in \( n_i \)). To examine the stability of the basic state \( n_i = 0 \) we put

\[
\begin{pmatrix}
n_1 \\
n_2
\end{pmatrix} = \begin{pmatrix}
q_1 \\
q_2
\end{pmatrix} e^{\lambda t} e^{ikx}
\]

and find from (9.11) the linear system

\[
(\tilde{L} - \lambda) \vec{q} = 0
\]

with the Jacobi matrix

\[
L = \begin{pmatrix}
a_1 - D_1 k^2, & a_2 \\
b_1, & b_2 - D_2 k^2
\end{pmatrix}
\]

From the solvability condition one determines

\[
\lambda = \lambda(k)
\]

the growth rate as a function of the wave number (or wave length) of the growing structures.

If \( a_2 \cdot b_1 < 0, b_2 < 0 \), there is a critical \( a_1 = a_c \), where \( \lambda(k_c) = 0 \)

After a little algebra it turns out that

\[
k_c^2 = \frac{1}{2} \left( \frac{a_1}{D_1} + \frac{b_2}{D_2} \right)
\]
For $a_1$ close to $a_c$, structures with wave length $\lambda \approx 2\pi/k_c$ grow! They are periodic in space and called

*Turing structures*

Turing structures were predicted by Alan Turing 1952 and first found experimentally by Castets et al. 1990. They used the so-called CIMA reaction, a chemical reaction based on chlorite, iodide and malonic acid.

Experimentally observed patterns (chemical concentrations, ph-values, etc.) by Ouyang and Swinney (1991):
The skin and coat patterns of certain animals can also be explained via Turing patterns. Basically, this was Turing’s original motivation to develop his theory, but it remains still a hypothesis.

**Application: skin, coat patterns**

- **pigment cells**
- **melanocytes**
- **chemical activators**
- **and inhibitors produce**
- **Turing-structure**
- **colors skin, hairs**

reaction is laid down in the embryonic development

*Thomas (1980), Murray (1981)*

*result:*

- Eigentliche Kaiserfisch
- Gemeine Argusfisch
9.4. PATTERN FORMATION
Cellular periodic patterns like hexagons, stripes or squares can be found in the famous convection experiments, done since 1900 by many different groups and authors all over the world.

Hexagons as a pattern of fluid motion on a free surface in a heated liquid, from Velarde et al.
Surface patterns of the temperature field, experiments by Schatz and Swinney
Chapter 10

The Kelvin-Helmholtz Instability

10.1 The system

- stratified layers, clouds, water waves (wind over water layer)
- instability of flat surface (old state)
- occurrence of traveling waves
- simple example for linear stability analysis
10.2 Mechanism and examples

10.2.1 Mechanism

Due to the different velocities of the two layers, the interface is unstable and folded.

Finally, a kind of spirals occur.
10.2.2 Examples

In cloud layers of different densities and velocities, with a little luck the Kelvin-Helmholtz instability can be seen.
10.3 Equations

We consider a fluid with no friction and assume that the velocity field in each layer can be derived from a potential $\Phi_i$. The starting point are the eqs. (6.2), formulated for each layer.

First, we transform them to a system moving with $\frac{U}{2} \hat{e}_x$ to the right.

\[
\vec{v}_1 = \frac{U}{2} \hat{e}_x + \nabla \Phi_1 \quad (10.1)
\]
\[
\vec{v}_2 = -\frac{U}{2} \hat{e}_x + \nabla \Phi_2
\]

Index 1 denotes the top layer (air), index two the bottom layer (water).

Inserting (10.1) into (6.2) yields

\[
\dot{\Phi}_1 = -\frac{p_1}{\rho_1} - gz - \frac{1}{2} U \partial_x \Phi_1 - \frac{1}{2} (\nabla \Phi_1)^2 \quad (10.2)
\]
\[
\dot{\Phi}_2 = -\frac{p_2}{\rho_2} - gz + \frac{1}{2} U \partial_x \Phi_2 - \frac{1}{2} (\nabla \Phi_2)^2 \quad (10.3)
\]

For the interface one has again the kinematic boundary condition

\[
\dot{h} = -\vec{v}_H \nabla h + v_z = -v_x \partial_x h + v_z \quad (10.4)
\]
10.4. THE LINEAR PROBLEM

This must be valid in both layers:

\[
h = \frac{-U}{2} \partial_x h - \partial_x \Phi_1 \cdot \partial_x h + \partial_z \Phi_1 \tag{10.5}
\]

\[
h = \frac{U}{2} \partial_x h - \partial_x \Phi_2 \cdot \partial_x h + \partial_z \Phi_2 \tag{10.6}
\]

### 10.4 The linear problem

The linear problem is given by neglecting all nonlinearities in the eqs. (10.2)-(10.6). To perform the linear stability analysis, one first has to eliminate the pressure. This is done by multiplying (10.2) with \( \rho_1 \), (10.3) with \( \rho_2 \) and subtracting them (both eqs. are evaluated at \( z = h \)):

\[
\rho_1 \Phi_1 - \rho_2 \Phi_2 = -\sigma \partial_{xx}^2 h + g \alpha h - \frac{1}{2} U \partial_x (\rho_1 \Phi_1 + \rho_2 \Phi_2) \tag{10.7}
\]

with \( \alpha = \rho_2 - \rho_1 \). In addition we have used

\[
p_2(z = 0) = p_1(z = 0) - \sigma \partial_{xx}^2 h
\]

where the last term is the Laplace pressure and accounts for the additional pressure from surface tension \( \sigma \), if the surface is not flat. For a flat surface one has simply

\[
p_2(z = 0) = p_1(z = 0)
\]

Linearization of (10.5) and (10.6) yields

\[
h = -\frac{U}{2} \partial_x h + \partial_z \Phi_1 \tag{10.8}
\]

\[
h = \frac{U}{2} \partial_x h + \partial_z \Phi_2 \tag{10.9}
\]

If both layers have an infinite depth (in reality, this means a large depth compared to the wave length of the surface structures, the linear system is solved by the ansatz (see sect. on gravity waves in deep water) :
\[ \Phi_1 = A_1 e^{-|k|z} e^{ikx} e^{\lambda t} \]
\[ \Phi_2 = A_2 e^{|k|z} e^{ikx} e^{\lambda t} \]  \hspace{1cm} (10.10)
\[ h = B e^{ikx} e^{\lambda t} \]

The form used for \( \Phi_i \) fulfills the Laplace equations

\[ \Delta \Phi_i = 0 \]

in each layer.

The velocity potentials exponentially decrease with the distance from the interface in each layer. Thus, the additional fluid motion described by \( \partial_z \Phi_i \) is concentrated to a region of width \( \sim 1/|k| \) around the interface (see figure). Outside that region, the fluid is more or less constantly moving with \( \pm U \).

Inserting (10.10) into (10.7), (10.8), and (10.9) leads to a linear homogeneous system (eigenvalue problem) for the amplitudes \( A_1, A_2, \) and \( B \) of the form

\[ \cdots \]
10.5. INSTABILITY

The uniform motion with a flat interface is unstable if one $\text{Re}(\lambda_i) > 0$. This is only possible if the square root in (10.13) is real, and therefore

$$R > 0$$
From that condition one finds an upper limit $U_c$ which must be exceeded by the relative velocity $U$:

$$U > U_c$$  \hspace{1cm} (10.16)

with

$$U_c = \sqrt{\frac{\rho_1 + \rho_2}{\rho_1 \rho_2} \left( \sigma |k| + \frac{g(\rho_2 - \rho_1)}{|k|} \right)}$$  \hspace{1cm} (10.17)

In summary, we found from the linear stability analysis

- there is a critical $U_c$
- if $U > U_c$ the flat interface is unstable
- at $U = U_c^{\text{min}}$ waves with $k = k_c$ become first unstable. From $dR/dk = 0$ one finds

$$k_c = \sqrt{\frac{g(\rho_1 - \rho_2)}{\sigma}}, \quad U_c^{\text{min}} = \sqrt{\frac{2\rho_1 + \rho_2}{\rho_1 \rho_2}} \sqrt{g \sigma (\rho_2 - \rho_1)}$$  \hspace{1cm} (10.18)

- traveling (or standing) waves grow exponentially in time if $U_c > U_c^{\text{min}}$

Above threshold, the interface has the form
10.6 Example: Water/Air – Layers

As an example, we compute the minimal critical velocity for a water/air interface that is necessary for the growth of the instability. Using

\[ \rho_1 = 1 \text{ kg/m}^3, \quad \rho_2 = 1000 \text{ kg/m}^3, \quad \sigma = 0.07 \text{ N/m}, \quad g = 9.81 \text{ m/s}^2 \]

one finds from (10.18)

\[ U_{c}^{\text{min}} \approx 7.2 \text{ m/s} \]

The wave number of the growing structure is

\[ k_c \approx 370 \text{ m}^{-1} \]

leading to a rather short wave length of

\[ \lambda_c = 2\pi/k_c \approx 1.7 \text{ cm} \]

On a windy day, it is sometimes possible to see such small ripples on the surface of lakes or large puddles.
Chapter 11

The Rayleigh-Bénard Instability

- Regular fluid patterns were first observed by Henri Bénard at the beginning of the 20th century in a uniformly heated oil layer. Looking on the top of the open surface of the fluid, he saw hexagons.

This is the original apparatus of Bénard (left), together with a pattern of hexagons on the surface (right):

- The instability is found in fluids which are heated from below. It is also named *convection instability*. 
• The notion convection means transport of material or “physical properties” (temperature, energy, ...) by (fluid) particles. Therefore, convection is only possible in fluids (liquids or gases) and not in solids.

• Convection instability is also found in micro gravity conditions (space experiments). Since there is now buoyancy, the instability must be triggered by another effect, the so-called Marangoni effect.

• The Marangoni effect is the (approximately linear) dependence of the surface tension of a liquid on temperature. This effect is also present on earth. In fact, the early experimental observations by Henri Bénard (1903) were based on the Marangoni effect rather than on buoyancy.

• Buoyancy driven instabilities can also be found in atmospheric motions.

![Image showing hexagonal motion of a fluid heated from below, found by a computer solution of the Navier-Stokes eqs. Shown are contour lines of the temperature field.](image)

The figure shows hexagonal motion of a fluid heated from below, found by a computer solution of the Navier-Stokes eqs. Shown are contour lines of the temperature field.

### 11.1 The system

We consider a fluid between two rigid (metal or glass) plates:
The plates are kept on constant temperatures $T_0$ (bottom) and $T_1$ (top). This defines the (negative) temperature gradient in vertical direction

$$\frac{T_0 - T_1}{d} = \beta$$

with $d$ as the distance between the two plates. The value of $\beta$ can be easily controlled from outside and is called *control parameter*.

The density of fluids depends on the temperature. If $\beta > 0$ the fluid is heated from below. Then there are two different mechanisms which may cause an instability of the fluid in rest:

- **Buoyancy**: Hot fluid particles (volume elements) near the bottom are lighter than colder ones and want to rise. Colder particles near the top want to sink. If the stabilizing forces of thermal conduction and friction in the fluid are exceeded by the externally applied temperature gradient, a (regular, spatially periodic) fluid motion sets in.

- **Surface tension**: If the upper surface of the fluid is free, i.e. in contact with the ambient air, tangential surface tension increases with decreasing surface temperature. If a fluid particle
near the surface moves by fluctuations say to the right, then warmer fluid is pulled up from
the bottom, increasing the surface temperature locally. Due to increasing laterally surface
tension with respect to the neighbored points, even more hot fluid is pumped up from the
bottom and the fluid starts to move. This is called the Marangoni effect and works even
without gravity, i.e. in space experiments.

In both cases, the typical length of the structures which bifurcate from the motionless state are
of the order of the layer depth. This instability is sometimes called short scale instability.

11.2 The basic equations

11.2.1 Navier-Stokes equations

For simplicity, we first treat the problem in 2D, $x$ (horizontal) and $z$ (vertical). The basic equations
for a viscous, incompressible fluid are

$$
\frac{\partial}{\partial x} v_x + \frac{\partial}{\partial z} v_z = 0 \quad (11.1)
$$

$$
\dot{v}_x + v_x \frac{\partial}{\partial x} v_x + v_z \frac{\partial}{\partial z} v_x = \nu \Delta v_x - \frac{1}{\rho_0} \frac{\partial}{\partial x} p \quad (11.2)
$$

$$
\dot{v}_z + v_x \frac{\partial}{\partial x} v_z + v_z \frac{\partial}{\partial z} v_z = \nu \Delta v_z - \frac{1}{\rho_0} \frac{\partial}{\partial z} p - \frac{\rho(T)}{\rho_0} g \quad (11.3)
$$

Here the Boussinesq approximation is used. This means all fluid parameters are considered to
be constant, except for the density in the gravity force term in (11.3). Taking the $z$ derivative of
(11.2) and subtracting the $x$ derivative of (11.3) eliminates the pressure and yields

$$
\frac{\partial}{\partial x} v_x - \frac{\partial}{\partial z} v_z + v_x \Delta v_x - v_z \Delta v_z = \nu \Delta (\frac{\partial}{\partial x} v_x - \frac{\partial}{\partial z} v_z) + \frac{g}{\rho_0} \frac{\partial}{\partial z} \rho(T) \quad (11.4)
$$

In 2D, one can always use a stream function, defined as:
11.2. THE BASIC EQUATIONS

\[ v_x = \partial_z \Phi(x, z, t), \quad v_z = -\partial_x \Phi(x, z, t) \]

Then, eq. (11.4) turns to

\[ \frac{\partial \Delta \Phi}{\partial t} - \partial_x \Phi \partial_z \Delta \Phi + \partial_z \Phi \partial_x \Delta \Phi = v \Delta^2 \Phi + \frac{g}{\rho_0} \partial_x g(T) \]  (11.5)

with the biharmonic operator

\[ \Delta^2 = \partial_{xxxx} + 2 \partial_{xx} \partial_{zz} + \partial_{zzzz} \]

11.2.2 Temperature equation

We wish to compute the change of temperature in a volume element that moves with \( \vec{v} \) (Lagrange description).

To do so, we apply the first law of thermodynamics to the volume element:

\[
\frac{dU}{dt} = \frac{dQ}{dt} + \frac{dW}{dt}
\]

inner energy = heat energy coming from work applied on \( \Delta V \)
outside \( = p \, dV = 0 \) (incompr. fluid)

Thus the inner energy changes with time according to

\[ \frac{dU}{dt} = \frac{dQ}{dt} \]  (11.6)
On the other hand one has the thermodynamic relation

\[ dU = \rho_0 c_v dT \]  

(11.7)

with the specific heat per volume \( c_v \). The right hand side of (11.6) can be expressed as sources or sinks of the heat current

\[ \vec{j} = -\lambda \text{grad} T \]

with the thermal conductivity \( \lambda \). Thus one has

\[ \frac{dQ}{dt} = -\text{div} \vec{j} = -\lambda \Delta T \]

and, using (11.6) with (11.7) finally

\[ \frac{dT}{dt} = \frac{\partial T}{\partial t} + \partial_x \Phi \partial_x T - \partial_z \Phi \partial_z T = \kappa \Delta T \]  

(11.8)

Here,

\[ \kappa = \frac{\lambda}{c_v \rho_0} \]

denotes the thermal diffusivity.

### 11.2.3 State equation

To close the system, we need the dependence of density on temperature, a state equation. As mentioned above, the Boussinesq approximation is applied, where a linear relation

\[ \rho(T) = \rho_0 (1 - \alpha(T - T_0)) \]  

(11.9)

in the buoyancy term is assumed. This defines

\[ \alpha = -\frac{1}{\rho_0} \frac{d\rho}{dT}, \]

the thermal expansion coefficient \( \alpha \).
To sum up, the state of the 2D-system is completely described by the two scalar fields

\[ \Phi(x, z, t), \quad T(x, z, t) \]

The material parameters are

\[ \rho_0, \nu, \kappa, \alpha \]

The control parameter is

\[ \beta = \frac{T_0 - T_1}{d} \]

### 11.2.4 Basic equations in dimensionless formulation

It is common to use the following scaling to dimensionless independent variables:

\[ x = x' \cdot d, \quad z = z' \cdot d, \quad t = \frac{d^2}{\kappa} t' \]

where \( d \) is the thickness of the fluid layer (distance between the two plates) and

\[ \frac{d^2}{\kappa} \equiv \tau \]

defines the “vertical diffusion time of heat”.

The dependent variables are scaled by

\[ \Phi = \kappa \cdot \Phi', \quad T = \beta \cdot d \cdot T' \]

Using this scaling, the basic eqs. (11.5) and (11.8) take the dimensionless form (we suppress all primes)

\[
\frac{1}{Pr} \left[ \frac{\partial \Delta \Phi}{\partial t} - \partial_x \Phi \partial_z \Delta \Phi + \partial_z \Phi \partial_x \Delta \Phi \right] = \Delta^2 \Phi - R \partial_x T \quad (11.10)
\]

\[
\frac{\partial T}{\partial t} + \partial_x \Phi \partial_x T - \partial_z \Phi \partial_z T = \Delta T \quad (11.11)
\]

Two dimensionless parameters (or “numbers”) are defined:
• The Rayleigh number.

\[ R = \frac{g\alpha \beta d^4}{\kappa \nu} \]

It is proportional to the temperature and can be considered as control parameter. It also depends on fluid properties and on the geometry.

• The Prandtl number.

\[ Pr = \frac{\nu}{\kappa} \]

The ratio between two diffusion constants (or times) depends only on the material. Small \( Pr \) indicates that heat diffusion processes are faster than momentum diffusion (gases, liquid metals), large \( Pr \) means momentum diffusion is the fast time scale (oils).

In addition one has to specify boundary conditions, what we shall do below.

11.3 The conductive state and its instability

The state where the fluid is in rest is a fixed point (stationary solution) of eqs. (11.10), (11.11). It is called conductive state because the heat is transported solely by heat conduction through the liquid. We shall first compute the temperature profile of this state.

11.3.1 The conductive state

One has (the index “0” denotes the conductive state)

\[ \vec{v}^0 = 0, \quad \phi^0 = 0 \]

everywhere and as a consequence from (11.10)

\[ \partial_x T^0 = 0, \quad T^0 = T^0(z) \]

From (11.11), only

\[ \Delta T^0(z) = \partial_{zz}^2 T^0 = 0 \]
is left which has the solution

\[ T^0(z) = a + bz \]

The integration constants are determined by the boundary conditions at \( z = 0 \) (bottom plate) and \( z = 1 \) (top) where the temperature is kept fixed:

\[ T^0(0) = \frac{T_0}{\beta \cdot d} = \frac{T_0}{T_0 - T_1}, \quad T^0(1) = \frac{T_1}{\beta \cdot d} = \frac{T_1}{T_0 - T_1} \]

Finally one has

\[ T^0(z) = \frac{T_0}{T_0 - T_1} - z \quad (11.12) \]

### 11.3.2 Linear stability analysis

To check the stability, we add small perturbations to the conductive state:

\[ T(x, z, t) = T^0(z) + \Theta(x, z, t), \quad \Phi(x, z, t) \]

Inserting this into (11.10), (11.11) and linearizing leads to the linear system

\[ \frac{1}{Pr} \frac{\partial \Delta \Phi}{\partial t} = \Delta^2 \Phi - R \frac{\partial}{\partial x} \Theta \quad (11.13) \]

\[ \frac{\partial \Theta}{\partial t} = \Delta \Theta - \frac{\partial}{\partial x} \Phi \quad (11.14) \]

Taking free boundary conditions for the velocity field at \( z = 0, 1 \)

\[ v_z = 0, \quad \frac{\partial^2}{\partial z^2} v_z = 0, \quad \Rightarrow \quad \Phi = 0, \quad \frac{\partial^2}{\partial z^2} \Phi = 0 \]

The disturbances of the temperature have to vanish at the plates

\[ \Theta(z = 0) = \Theta(z = 1) = 0 \]
Thus one may solve (11.13), (11.14) by

\[ \Theta(x, z, t) = A e^{ikx} e^{\lambda t} \sin \ell \pi z \quad (11.15) \]

\[ \Phi(x, z, t) = B e^{ikx} e^{\lambda t} \sin \ell \pi z, \quad \ell = 1, 2, \ldots \quad (11.16) \]

leading to a linear eigenvalue problem for \( A \) and \( B \). From its solvability condition one finds the eigenvalues (we assume infinite \( Pr \), neglecting terms with \( 1/Pr \)):

\[
\lambda_{\ell}(k) = -k^2 - \pi^2 \ell^2 + \frac{Rk^2}{(k^2 + \pi^2 \ell^2)^2} \quad (11.17)
\]

### 11.3.3 Linear growth rates and modes

The growth rate (11.17) is a function of both the horizontal wave number \( k \) and the number of nodes in vertical direction \( \ell \). The following figure shows the growth rate for different values of \( k \) and \( \ell \).

Clearly, modes with \( \ell = 1 \) have the largest growth rate and become first unstable. This can also be seen by computing the so-called marginal (or critical) line by putting (11.17) to zero and solving it for \( R \):

\[ R_c(k) = \frac{(k^2 + \pi^2 \ell^2)^3}{k^2} \]
The marginal line is sketched in the figure above, together with some typical modes. Increasing the temperature gradient and therewith the Rayleigh number, convection sets first in where $R(k)$ has its minimum,

$$R_c^\min = \frac{27}{4} \pi^4 \approx 657.5$$

with the wave number

$$k_c = \frac{\pi}{\sqrt{2}} \approx 2.22$$

For realistic, no-slip boundary conditions along the plates, the linear problem (11.13), (11.14) can only be solved numerically, for instance by a Galerkin method. The qualitative form of $R_c(k)$ is the same, but the critical values are different. One finds

$$R_c^\min = 1707.7 \ , \quad k_c = 3.12$$
11.4 The fully nonlinear equations – Numerical results

In this section we present some numerical results of the 3D basic equations.

11.4.1 The basic equations in three dimensions

In three dimensions, the concept of a stream function does not apply. Instead one may decompose
the velocity field in its so-called toroidal and poloidal parts:

\[
\vec{v}(\vec{r}, t) = \vec{v}_T(\vec{r}, t) + \vec{v}_P(\vec{r}, t) .
\] (11.18)

Each part can be expressed by a single scalar function \(\Phi\), \(\Psi\) respectively:

\[
\vec{v}_T = \nabla \times (\Phi \hat{e}_z) = \begin{pmatrix}
\partial_y \Phi \\
-\partial_x \Phi \\
0
\end{pmatrix},
\vec{v}_P = \nabla \times \nabla \times (\Psi \hat{e}_z) = \begin{pmatrix}
\partial_z \partial_x \Psi \\
\partial_z \partial_y \Psi \\
-\Delta_2 \Psi
\end{pmatrix}
\] (11.19)

with \(\Delta_2 = \partial^2_{xx} + \partial^2_{yy}\). Here, \(\Phi\) represents a kind of stream function for the toroidal part of the velocity field (\(\vec{v}_T\) is only 2D, it has no vertical component), \(\Psi\) can be considered as a generalized potential for the poloidal field.

Taking the curl of the Navier-Stokes eqs. its z-component reads

\[
\left\{ \Delta - \frac{1}{Pr} \partial_t \right\} \Delta_2 \Phi(\vec{r}, t) = - \frac{1}{Pr} \left[ \nabla \times ((\vec{v} \cdot \nabla) \vec{v}) \right]_z .
\] (11.20)

For sake of clarity we wrote \(\vec{v}\) on the left hand side, which has to be substituted by (11.18, 11.19). An equation for \(\Psi\) is found by taking twice the curl of the Navier-Stokes eqs.:

\[
\left\{ \Delta - \frac{1}{Pr} \partial_t \right\} \Delta \Delta_2 \Psi(\vec{r}, t) = - R \Delta_2 \Theta(\vec{r}, t) - \frac{1}{Pr} \left[ \nabla \times \nabla \times ((\vec{v} \cdot \nabla) \vec{v}) \right]_z .
\] (11.21)

The equation for the temperature has the same form than (11.11) which we repeat here for completeness:

\[
\left\{ \Delta - \partial_t \right\} \Theta(\vec{r}, t) = - \Delta_2 \Psi(\vec{r}, t) + (\vec{v} \cdot \nabla) \Theta(\vec{r}, t) .
\] (11.22)
The equations (11.20), (11.21), and (11.22) form the basic system for the three variables $\Phi$, $\Psi$, and $\Theta$. They must be completed by boundary conditions.

## 11.4.2 Boundary conditions

Since we wish to examine laterally extended systems, we can use periodic boundary conditions for $x, y$ with a large periodicity length (aspect ratio) compared to $d$. In vertical direction, the situation deserves more attention.

### A. Bottom plate ($z = 0$)

For viscous fluids, all velocity components have to vanish at a solid wall (bottom plate):

$$\vec{v}\Big|_{z=0} = 0 .$$ (11.23)

These are the no-slip conditions. In terms of $\Phi$ and $\Psi$ they read

$$\Psi\Big|_{z=0} = \Phi\Big|_{z=0} = 0 .$$ (11.24)

(In fact a constant would also do it, but this can be put to zero). For vanishing $\vec{v}_P$ one has to require in addition

$$\partial_z \Psi\Big|_{z=0} = 0 .$$ (11.25)

The temperature on the bottom is given by $T_0$, so the deviation has to vanish:

$$\Theta\Big|_{z=0} = 0 .$$ (11.26)

### B. Surface of the fluid ($z = 1$)

If the surface is closed by a rigid plate, the same conditions (11.24) - (11.26) hold at $z = 1$.

On a free, flat surface with normal $\hat{e}_z$, only the vertical velocity component has to vanish. In addition, the $xz$ and $yz$ components of the viscous stress tensor have to be balanced by surface stress forces:
\[ v_z|_{z=1} = 0 \] (11.27a)
\[ \eta \partial_x v_x|_{z=1} = \partial_x \Gamma|_{z=1} \] (11.27b)
\[ \eta \partial_y v_y|_{z=1} = \partial_y \Gamma|_{z=1} , \] (11.27c)

with the surface tension \( \Gamma \). Differentiating (11.27b) with respect to \( x \), (11.27c) with respect to \( y \) and adding both equations yields with the continuity eq.

\[ \eta \partial_{zz} v_z|_{z=1} = -\Delta \Gamma|_{z=1} . \] (11.28)

Spatial surface tension inhomogeneities may be caused by a temperature gradient. To include the Marangoni effect one assumes that the surface tension is a linear function of temperature:

\[ \Gamma(T) \approx \Gamma(T_0) - \gamma_T \cdot (T - T_0) . \] (11.29)

Here

\[ \gamma_T = -\left. \frac{d\Gamma}{dT} \right|_{T=T_0} \]

denotes the dependence of surface tension on temperature. For fluids, one has \( \gamma_T > 0 \). Inserting this and the velocity decomposition (11.10) into (11.28) gives after scaling the boundary conditions for \( \Psi \):

\[ \partial_{zz} \Psi|_{z=1} = Ma \cdot \Theta|_{z=1} \] (11.30)

with the non-dimensional \textit{Marangoni number} defined as

\[ Ma = \frac{\gamma_T \beta d^2}{\kappa \eta} = \frac{\gamma_T \beta d^2}{\rho \kappa V} . \] (11.31)

For \( \Phi \) one finds from the \( z \)-component of the curl of (11.27)

\[ \partial_z \Phi|_{z=1} = 0 . \] (11.32)

The last boundary condition we need is that for the temperature. If the surface is not a good thermal conductor one can derive it by solving the heat equation in the gas layer above the fluid. If motion is neglected in this layer, one finds
with the non-dimensional (positive) Biot number $Bi$ which describes the ratio of thermal conduction of the boundary (here the gas or atmosphere) to that of the fluid. For perfectly conducting boundaries $Bi \rightarrow \infty$, for bad conductors, $Bi << 1$.

### 11.4.3 Results

**A. Closed upper surface**

If the surface is closed by a good conductor, rolls are the preferred structure close to onset:

Depending on $Pr$, these rolls are either more or less parallel (with some possible defects and grain boundaries) or have the tendency to form spirals. The bending and formation of spirals is caused by the so-called mean flow, this is the horizontal large scale motion of the fluid which is presented by the stream function $\Phi$. 
Slightly bended rolls produce a mean flow (a) which further increases the deformation of the rolls (b). This kind of instability may lead to a break up of the rolls and the formation of spirals (c).

Spirals are found for small Prandtl number (in compressed gases) and not too far above threshold. The next figure shows a numerically found time series.
Increasing $R$ further, spirals become again unstable and a turbulent structure is found for small $Pr$ and larger Rayleigh number $Pr = 0.2, R = 2R_c$. In contrast to fully developed turbulence, a typical length scale is still present. We complete this part by presenting a time dependent, weakly turbulent series for small $Pr$ relatively far above threshold.
B. Free upper surface

If the Marangoni effect is the main driving mechanism, hexagons are obtained at threshold as already seen. Defects may travel through the structure and survive for a long time, even in high Pr fluids.

Penta-hepta defects are marked by black dots. They travel slowly in lateral direction.
A secondary instability takes place to squares, if the heating is increased. The threshold of this instability increases with $Pr$.

For even larger values of $Ma$, time dependent structures emerge:
Further reading

The field of hydrodynamics is huge and still growing fast. The script can contain only basic knowledge and simple or simplified applications. For further reading we recommend the following:


(mainly for Part I)

For Part II and III we refer to


**T. E. Faber,** *Fluid Dynamics for Physicists*, Cambridge University Press.

For those who like mathematics, we mention the short monograph

**A. J. Chorin and J. E. Marsden,** *A Mathematical Introduction to Fluid Mechanics*, Springer-Verlag

Many applications can be found in

**Schaum’s Outlines, Fluid Dynamics,** Mc Graw-Hill

On Solitons (chapt. 8), we recommend


Nice pictures but now formulas are in

**M. Van Dyke, An Album of Fluid Motion,** Parabolic Press

Old, but still good (mainly for Part II)


Convection instability (Chapt. 11) is explained in detail in the first chapters of

Pattern formation in general can be found in the books of Haken:

H. Haken, *Synergetics. An Introduction*, Springer-Verlag


A detailed compilation on pattern formation with many further references is


For numerical methods see e.g.


and finally, unfortunately up to now only in german,

M. Bestehorn, *Hydrodynamik und Strukturbildung*, Springer-Verlag
Appendix A

Exercises

A.1 Sheet 1

Problem 1: Vector field

A. Consider the vector field \( \mathbf{v} = x^3 \mathbf{e}_1 + z^3 \mathbf{e}_2 + y^3 \mathbf{e}_3 \). Determine

a) \( (\mathbf{v} \cdot \nabla) \mathbf{v} \)

b) \( \text{div} \mathbf{v} \) and \( \text{curl} \mathbf{v} \)

in cartesian coordinates at the position \( P = P(2, -1, 0) \). (2 points)

B. Show that

a) \( \text{curl} (\text{grad} U) = 0 \), for every scalar field \( U \)

b) \( \text{grad}(\phi \psi) = \phi \text{grad} \psi + \psi \text{grad} \phi \), for every scalar fields \( \phi, \psi \)

c) \( \text{div}(\phi \mathbf{A}) = \phi \text{div} \mathbf{A} + \mathbf{A} \cdot \text{grad} \phi \), for every scalar field \( \phi \) and vector field \( \mathbf{A} \).

(3 points)
Problem 2: Eigenvalues and eigenvectors

The strain tensor \( \varepsilon_{ij} \) is related to the displacement field \( \vec{s} \) by

\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial s_i}{\partial x_j} + \frac{\partial s_j}{\partial x_i} \right).
\]

For a two-dimensional parallel shearing the vector \( \vec{s} \) is given by:

\[
\vec{s} = \begin{pmatrix}
s_1(x_2) \\
0
\end{pmatrix}.
\]

Find the principal axes. (2 points)

Problem 3: Stress tensor

A. The state of stress at a point is given by the stress tensor

\[
\sigma_{ij} = \sigma_0 \begin{pmatrix}
1 & a & b \\
a & 1 & c \\
b & c & 1
\end{pmatrix}.
\]

where \( a, b, c \) are constants and \( \sigma_0 \) is some stress value. Determine the constants \( a, b \) and \( c \) so that the stress tensor on the plane \( \vec{n} = \frac{1}{\sqrt{3}} \vec{e}_1 + \frac{1}{\sqrt{3}} \vec{e}_2 + \frac{1}{\sqrt{3}} \vec{e}_3 \) vanishes.

(1 point)

B. Let the stress components (in MPa) at point P with respect to the system \( S \) be expressed by the matrix

\[
\sigma_{ij} = \begin{pmatrix}
2 & -2 & 0 \\
-2 & \sqrt{2} & 0 \\
0 & 0 & -\sqrt{2}
\end{pmatrix}
\]

and let the primed system \( S' \) obtained by a 45\(^\circ\) counterclockwise rotation about the \( x_3 \) axis. Determine the stress components \( \sigma'_{ij} \). (2 points)
Problem 1: Hydrostatics

Figure below shows a manometer, which is a U-shaped tube containing mercury of density $\rho_m$. Manometers are used as pressure measuring devices. If the fluid in the tank A has a pressure $p$ and density $\rho$, then show that the gauge pressure in the tank is

$$p - p_{atm} = \rho_m gh - \rho ga.$$

Note that the last term on the right-hand is negligible if $\rho << \rho_m$.

(*Hint: Equate the pressure at X and Y in the above figure.*) (1 point)

Problem 2: Hydrostatics or Euler equation

A. What is the value of $a_x$ at which the water just begins to run out over the rear wall of the container shown in figure below? (3 points)
B. A cylinder filled with two immiscible liquids of density $\rho_1$ and $\rho_2 > \rho_1$ rotates with angular velocity $\omega$ about its axis. Find the pressure distribution, the shape of the interface between the liquids, and the shape of the free surface.

(*Hint:* In a rotating frame of reference one has to consider two supplementary forces on the right side of Euler equation. One is the Coriolis force $-2\hat{\omega} \times \vec{v}$ and the second one is the centrifugal force $+\omega^2 \vec{r}$, where $\vec{r}$ is the vector of position drawn perpendicularly to the axis of rotation.)(4 points)

---

**Problem 3: Continuity equation**

Let a one-dimensional velocity field be $v_x = v_x(x, t)$, with $v_y = 0$ and $v_z = 0$. The density varies as $\rho = \rho_0(2 - \cos \omega t)$. Find an expression for $v_x(x, t)$ if $v_x(0, t) = V_0$. (2 points)
A.3 Sheet 3

Problem 1: Bernoulli equation

Figure below shows a simple device to measure the local velocity in a fluid stream by inserting a narrow bent tube. This device is called a Pitot tube, after the French mathematician Henry Pitot (1695-1771), who used a bent glass tube to measure the velocity of the river Seine. Considering two points 1 and 2 at the same level (point 1 away from the tube and point 2 immediately in the front of the open end where the fluid velocity is zero) and using Bernoulli equation, compute the fluid velocity. There are known the heights $h_1$, $h_2$, the fluid density $\rho$ and the gravity constant $g$. Friction is negligible along the streamlines. Depends the “measured velocity” on the fluid density? (2 points)

Problem 2: Barometric formula

A. Expressions for the pressure distribution and “thickness” of the atmosphere can be obtained by assuming that they are isothermal. This is a good assumption in a lower 70km of the atmosphere, where the absolute temperature remains within 15% of 250 K. Using the formula which describes the pressure in a static fluid

$$\frac{dp}{dz} = -\rho g$$  \hspace{1cm} (A.1)
and the equation of the state for an ideal gas

\[ p = \frac{\rho RT}{\mu} \quad (A.2) \]

show that

\[ p = p_0 e^{-\frac{\mu gz}{RT}} \quad (A.3) \]

where \( p_0 \) is the pressure at \( z = 0 \), \( \mu \) is the molar mass, \( g \) is gravity and \( R \) is the universal gas constant. (2 points)

B. The quantity \( RT/\mu g \), called the scale height, is a good measure of the thickness of the atmosphere. Evaluate the scale height for an average atmospheric temperature of \( T = 250 K \) \((\mu = 29 \text{ kg/kmol}, g = 9.8 \text{ N/kg}, R = 8.31 \cdot 10^3 \text{ J/kmol·K})\). Compute the total mass of the atmosphere. (The radius of the Earth is \( R_E = 6400 \text{ km} \) and the atmospheric pressure at \( z = 0 \) and \( T = 250 K \) is \( p_0 = 10^5 \text{ N/m}^2 \).) (2 points)

C. Assume now that the temperature of the atmosphere varies with the height \( z \) as

\[ T = T_0 + Kz. \]

Show that the pressure varies with the height as

\[ p = p_0 \left[ \frac{T_0}{T_0 + Kz} \right]^{\mu g/KR}. \quad (A.4) \]

(2 points)

D. For a real gas of van der Waals type the equation of the state becomes:

\[ p(\rho) = \frac{B\rho}{1 - b\rho} - A\rho^2 \quad (A.5) \]

where \( B, b, A \) are two coefficients taken as empirical parameters. Assuming an isothermal atmosphere one can write:

\[ dp = \left( \frac{\partial p}{\partial \rho} \right)_T d\rho \]

and therefore from (1) one obtains immediately:

\[ \int_{\rho(0)}^{\rho(z)} \frac{1}{\rho} \left( \frac{\partial p}{\partial \rho} \right)_T d\rho = -gz \quad (A.6) \]
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Integrating the relation (6) show that:

\[ z = -\frac{1}{g} \left[ B \left( \ln \rho - \ln (1 - b\rho) + \frac{1}{1 - b\rho} \right) - 2A\rho \right] + C \]  

(A.7)

with \( C \) the integration constant. (2 points)

E. Plot the coordinate \( z \) from relation (7) versus the density \( \rho \) for \( A=B=C=10 \), \( g=10 \) and \( b=0.5 \), in the range \( \rho = 0.00001 \) and \( \rho = 1.99 \). Indicate on this figure the position of the interface between the liquid and the gas phases. (1 point)
Problem 1: Velocity profile of inclined fluid layer

A layer of fluid with thickness $d$ is bounded above by a free and non-deformable surface and below by a fixed plane inclined at an angle $\alpha$ to the horizontal. Gravity (with acceleration constant $g$) induces a stream in $x$-direction. We search the stationary velocity profile and the pressure distribution in the fluid layer.

Use the 2D cartesian coordinate system from figure below.

The incompressible Navier-Stokes equation is given by

$$\frac{d\vec{v}}{dt} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{v} + \vec{f} \tag{A.8}$$

with the velocity $\vec{v} = (v_x, v_z)$, kinematic viscosity $\nu$ and $\vec{f}$ the body force density.

a) Show that Eq. (A.8) can be reduced on components to

$$0 = \nu \frac{d^2 v_x}{dz^2} + g \sin \alpha \tag{A.9}$$

$$0 = \frac{dp}{dz} + \rho g \cos \alpha \tag{A.10}$$

(2 points)

b) At the free non-deformable surface ($z = d$) we must have $\sigma_{xz} = \eta \frac{dv_x}{dz} = 0$ and for the pressure, we have $p = p_0$, $p_0$ being the atmospheric pressure. For $z = 0$ we have: $v_x(z = 0) = 0$. Using these boundary conditions integrate Eqs. (2) and (3) and show that:

$$p(z) = p_0 + \rho g (d - z) \cos \alpha \tag{A.11}$$

$$v_x(z) = \frac{g \sin \alpha}{2\nu} z(2d - z) \tag{A.12}$$
(2 points)

c) Plot the relations (4) and (5) versus $z$ in the range 0 and $d$. (1 point)

Problem 2: Couette flow between rotating cylinders

Useful formulas: Cylindrical polar coordinates are denoted by $(r, \phi, z)$, with $(v_r, v_\phi, v_z)$ the corresponding velocity components. In the cylindrical coordinates the continuity equation and Navier-Stokes equation (written here without body forces) become, respectively:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(\rho rv_r) + \frac{1}{r} \frac{\partial}{\partial \phi}(\rho v_\phi) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

$$\frac{\partial v_r}{\partial t} + (\vec{v} \cdot \nabla)v_r - \frac{v_\phi^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \Delta v_r - \frac{v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\phi}{\partial \phi} \right)$$

$$\frac{\partial v_\phi}{\partial t} + (\vec{v} \cdot \nabla)v_\phi + \frac{v_r v_\phi}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \phi} + \nu \left( \Delta v_\phi - \frac{v_\phi}{r^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \phi} \right)$$

$$\frac{\partial v_z}{\partial t} + (\vec{v} \cdot \nabla)v_z = -\frac{1}{\rho \partial z} + \nu \Delta v_z,$$

where

$$\vec{v} \cdot \nabla = v_r \frac{\partial}{\partial r} + \frac{v_\phi}{r} \frac{\partial}{\partial \phi} + v_z \frac{\partial}{\partial z},$$

$$\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}.$$

a) Determine the distribution of velocity in a viscous, incompressible fluid, existent between two infinite, coaxial cylinders, of radii $R_1$ and $R_2 > R_1$, which perform a uniform motion of rotation about their common axis, with angular velocities $\omega_1$, and $\omega_2$, respectively. (1 point)

b) Particularize the result from the point a) for $\omega_1 = \omega_2 = \omega$ and for the case when the exterior cylinder is taken away ($R_2 = \infty$, $\omega_2 = 0$). (1 point)

c) Determine the pressure distribution knowing the fluid density $\rho$ and the fluid pressure at the cylinder of radius $R_1$, $p_0$. (1 point)

Hint 1: Choose a cylindric system of coordinates, with $z$ axis along the cylinders axis. By symmetry
criteria we have: \( v_r = v_z = 0; \) \( v_\phi = v_\phi(r); \) \( p = p(r) \). Project the Navier-Stokes equation on \( r \) and \( \phi \) directions.

Hint 2: Equations of the type:

\[
\frac{d^2 v}{dr^2} + \frac{1}{r} \frac{dv}{dr} - \frac{v}{r^2} = 0
\]

has solutions of the form \( r^n \); substitution gives \( n = \pm 1 \), so that \( v(r) = Ar + B/r \).

**Problem 3: Streamlines, incompressibility and vorticity**

A two-dimensional steady flow has the velocity components: \( v_x = y, \) \( v_y = x \). Show that the streamlines are rectangular hyperbolas: \( x^2 - y^2 = \text{const} \). Indicate if this flow is compressible or incompressible? Do it has vorticity? (2 points)
A.5 Sheet 5

Problem 1: Sound waves in moving air

Find the dispersion equation for a plane sound wave in moving air, starting from the following system equations:

\[\rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} \right) = -\frac{\partial p}{\partial x}\]

\[\frac{\partial p}{\partial t} + \frac{\partial (\rho v_x)}{\partial x} = 0\]

\[p \rho^{-\gamma} = p_0 \rho_0^{-\gamma}\]

with \(\gamma\)–the adiabatic index. The air is moving at speed \(U_0\) in the positive \(x\) axis. Assume the wave propagation in \(x\) direction. (3 points)

Problem 2: Sound waves in the presence of heat conduction

Find the dispersion relation for plane sound waves in air, accounting for heat conduction. The governing equations are motion, continuity, state, and energy:

\[\rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} \right) = -\frac{\partial p}{\partial x}\]

\[\frac{\partial p}{\partial t} + \frac{\partial (\rho v_x)}{\partial x} = 0\]

\[p = \frac{\rho RT}{m_0}\]

\[\rho c_p \left( \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} \right) = \frac{\partial p}{\partial t} + v_x \frac{\partial p}{\partial x} + \kappa \frac{\partial^2 T}{\partial x^2}\]

with \(c_p\)–the specific heat capacity at constant pressure, \(\kappa\)–the thermal conductivity and \(m_0\)–the molar mass. The undisturbed state of the system is described by the parameters: \(v_x = 0, p = p_0 = const., \rho = \rho_0 = const., T = T_0 = const..\) We assume now small fluid perturbations depending on \(x\) and \(t\), of \(e^{i(kx-\omega t)}\) type, propagating along \(x\) axis. Show that:

\[\frac{\kappa}{\rho_0 c_p} k^2 \left( k^2 - \frac{\omega^2 \rho_0}{p_0} \right) = i\omega \left( k^2 - \frac{\omega^2 \rho_0}{\gamma p_0} \right)\]

with \(\gamma\)–the adiabatic index, \(\gamma = \frac{c_p}{c_v} = \frac{c_p}{c_p - R/m_0}\), \(c_v\)–the specific heat capacity at constant volume. (4 points)
Problem 3: Shallow water waves in a channel with linearly modulated ground

The linearized shallow water equation can be written:

$$\partial_{tt}h(x,t) - G\partial_x(H(x)\partial_x h(x,t)) = 0$$  \hspace{1cm} (A.13)

with $H(x)$ the real depth of the flat water (see lecture).

One considers a channel of length $L$ with linearly increasing depth

$$H(x) = x/L \hspace{1cm} 0 \leq x \leq L.$$

At $x = L$ the channel is in contact with the water from a lake. The water surface oscillates periodically in time with the frequency $\Omega$. One assumes the amplitudes of the water waves small enough, so that one can use Eq. (1). Derive the nonsingular solution for the water waves in the channel:

$$h(x,t) = 1 + a_0 J_0(\sqrt{k}x)\cos(\Omega t),$$  \hspace{1cm} (A.14)

with $k = 4\Omega^2 L/G$ and $a_0$ a small amplitude $|a_0| << 1. \hspace{0.5cm} (3 \text{ points})$

**Hint1:** In order to satisfy the boundary condition at $x = L$ use for Eq. 1 an ansatz of the form:

$$h(x,t) = 1 + a(x)\cos(\Omega t).$$

**Hint2:** Make the substitution: $x = \frac{G}{4\Omega^2 L^2} \xi^2$.

**Hint3:** A nonsingular solution for the equations of the type

$$\frac{d^2a}{d\xi^2} + \frac{1}{\xi} \frac{da}{d\xi} + a = 0$$

is a Bessel function of the first kind $J_0(\xi)$. 
A.6 Sheet 6

Problem 1: Ekman layer

The winds that blow over the Earth’s ocean give rise to surface currents, especially when they blow steadily in a fixed direction. In this case the ocean may approach a steady state of motion, such that, in a frame of reference which rotates with the Earth, we can have a balance between the Coriolis and the viscous forces in a frictional layer near the free surface of the ocean (called Ekman layer after the name of the Swedish oceanographer Ekman which worked out on this problem in 1905). We consider the frictional layer near the free surface of the ocean, which is acted by a wind stress $\tau$ in the $x$ direction. We examine the steady solution and we assume that the horizontal pressure gradients are zero (the pressure does not depend on $x$ and $y$) and the laminar flow in horizontal direction depends only on $z$.

(a) Starting from the Navier-Stokes equation for rotating frames of reference show that the horizontal equations of motion are

$$-f v_y = \nu \frac{d^2 v_x}{dz^2}$$  \hspace{1cm} (A.15)

$$f v_x = \nu \frac{d^2 v_y}{dz^2}$$  \hspace{1cm} (A.16)

with $f = 2 \Omega \sin \lambda$ and $\lambda$—the angle of the latitude measured from the equator in northerly direction (i.e. $\lambda = +\pi/2$ at the north pole, and $\lambda = -\pi/2$ at the south pole). (1 point)

(b) Taking the $z$ axis vertically upwards from the surface of the ocean, the boundary conditions are $\rho v \frac{dv_x}{dz} = \tau$, $\frac{dv_y}{dz} = 0$ at $z = 0$ and $v_x, v_z \to 0$ as $z \to -\infty$. Show that the velocity components become

$$v_x = \frac{\tau}{\rho} \frac{1}{\sqrt{f \nu}} \exp(z/\delta) \cos \left( -\frac{z}{\delta} + \frac{\pi}{4} \right)$$

$$v_y = -\frac{\tau}{\rho} \frac{1}{\sqrt{f \nu}} \exp(z/\delta) \sin \left( -\frac{z}{\delta} + \frac{\pi}{4} \right)$$

with $\delta = \sqrt{2f/\nu}$—the thickness of the Ekman layer. (2 points)

(c) Plot the velocities at various depths and the vertical distributions of $v_x$ and $v_z$ versus $z$ for the northern hemisphere. How will change these representations in the southern hemisphere? (2 points)

Problem 2: Foucault pendulum

A Foucault pendulum (named after the French physicist Léon Foucault) was conceived in 1851 as an experiment to demonstrate the rotation of the Earth without any other astronomical observations. This is a spherical pendulum subject to both gravitational and Coriolis forces.
(a) Show that the motion of the pendulum in the \((x, y)\) horizontal plane is described by the coupled equations (for small amplitudes):

\[
\frac{d^2 x}{dt^2} + \omega^2 x = -f \frac{dy}{dt}
\]  

(A.17)

\[
\frac{d^2 y}{dt^2} + \omega^2 y = f \frac{dx}{dt}
\]  

(A.18)

with \(\omega^2 = g/l\) (\(l\)—the length of the pendulum) and \(f = 2 \Omega \sin \lambda\), as in the Problem 1. (2 points)

(b) Show that

\[
\frac{d^4 x}{dt^4} + (2\omega^2 + f^2) \frac{d^2 x}{dt^2} + \omega^4 x = 0
\]  

(A.19)

Using for the above equation an ansatz of the form: \(x \sim \sin(\Omega t)\), show that in the limit \(f << \omega\) one obtains: \(\Omega_{1,2} = \omega \pm \frac{f}{\omega}\). (1 point)

(c) The most general solution of equation (5) is:

\[
x(t) = A \sin(\Omega_1 t + \phi_1) + B \sin(\Omega_2 t + \phi_2).
\]

Show that:

\[
y(t) = \frac{1}{\omega} \left[ -A \Omega_1 \cos(\Omega_1 t + \phi_1) + B \Omega_2 \cos(\Omega_2 t + \phi_2) \right].
\]  

(A.20)

(1 point)

(d) Considering that at the initial moment the pendulum oscillates in the vertical plane, so that \(A \Omega_1 = B \Omega_2\), find from the relation (6) the period of revolution of the Foucault pendulum. Depends this period on latitude? On the length of the pendulum? (1 point)

**Problem 3: Influence of rotation on atmospheric waves**

Consider the atmospheric perturbations described by the system equations:

\[
\rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = -\nabla p - 2\rho \vec{\Omega} \times \vec{v}
\]

\[
\frac{\partial p}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
\]

\[
p \rho^{-\gamma} = p_0 \rho_0^{-\gamma}
\]

where \(\vec{\Omega}\) is the Earth’s angular velocity assumed to be constant. One considers \(\vec{\Omega} = \begin{pmatrix} 0 \\ 0 \\ \Omega_\varepsilon \end{pmatrix}\) and \(\vec{v}(x, t) = \begin{pmatrix} v_x(x, t) \\ v_y(x, t) \\ 0 \end{pmatrix}\). We suppose the undisturbed state described by the parameters: \(v_x = 0, v_y = 0, \rho = \rho_0 = \)
Assuming now small fluid perturbations depending on $x$ and $t$, of $e^{(kx-\omega t)}$ type, propagating along $x$ axis, show that the system dispersion equation takes the form:

$$\omega^2 = k^2 c_s^2 + 4\Omega_c^2$$

with $c_s = \sqrt{\frac{T_0}{\rho_0}}$ — the sonic speed. Are the atmospheric waves dispersive or non-dispersive in the presence of rotation effects? (5 points)
Problem 1: One-soliton-solution

The Korteweg-de Vries equation can be written
\[ h + h \cdot \partial_t h + \partial^3_{xxx} h = 0 \] (A.21)
(see lecture). Derive the “One-soliton-solution”
\[ h(\xi) = \frac{3v}{\cosh^2(\sqrt{v}/2)} \] (A.22)
from Eq. 1. To do so, search for a solution traveling with velocity \( v \), i.e. use the transformation
\[ h(x, t) = h(x - vt) = h(\xi) \]
and integrate Eq. 1 two times. Sketch the solution (2) for \( v = 5; 10 \) and 15, respectively, in the range \( \xi = -10 \) and \( \xi = 10 \). Depends the soliton amplitude on the propagation velocity \( v \)? How? (3 points)

Problem 2: Sine-Gordon equation

The Korteweg-de Vries equation is not the only equation which admits soliton solutions. Another example is the Sine-Gordon equation which in one-dimensional form reads:
\[ c^2 \partial^2_{xx} - \partial^2_{tt} \phi(x, t) = \sin \phi(x, t). \] (A.23)

(a) For Sine-Gordon equation written above we look for solutions traveling with constant velocity \( v \), i.e. we use again the transformation
\[ \phi(x, t) = \phi(x - vt) = \phi(\xi). \]
Integrating once Eq. 3, show that:
\[ \frac{1}{2} (c^2 - v^2) \left( \frac{\partial \phi}{\partial \xi} \right)^2 = E - \cos \phi \] (A.24)
where \( E \) is an integration constant. (2 points)

(b) For solitary wave the solution \( \phi \) and \( \partial \phi / \partial \xi \) have to approach 0 as \( \xi \to \pm \infty \). Find under these conditions
the integration constant $E$. (0.5 points)

(c) Integrate the relation (4) with the integration constant computed above. Derive the soliton solution:

$$\phi(x,t) = 4 \arctan \left[ \exp \left( \pm \frac{x-x_0-vt}{\sqrt{c^2-v^2}} \right) \right],$$  \hspace{1cm} (A.25)

assuming the initial condition $\phi(x_0,0) = \pi$. (1 point)

(d) The solution with $(+)$ in the relation (5) gives the soliton (kink) solution, the solution with $(-)$ describes the antisoliton (antikink) solution. Plot the two solutions versus $x$ for $x_0 = 0, \ c = 10, \ v = 7$ at $t = 0$. What happen with the previous plots when $x_0$ changes from $x_0 = 0$ to $x_0 = 20$? (1.5 points)

Problem 3: Two-soliton-solution

One can verify that

$$h(x,t) = -12 \frac{3 + 4 \cosh(2x-8t) + \cosh(4x-64t)}{[3 \cosh(x-28t) + \cosh(3x-36t)]^2}$$  \hspace{1cm} (A.26)

is an exact solution of the modified Korteweg-de Vries equation:

$$\dot{h} - 6h \partial_t h + \partial^3_{xx} h = 0.$$

Plot the solution (6) versus $x$ in the range $x = -10$ and $x = 10$ at different moments of time: $t = -0.5$; $t = -0.1$; $t = 0$; $t = 0.1$; $t = 0.5$. Discuss the results. Explain why this solution is named a “two-soliton solution”. (2 points)